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Unified Maxwell-Einstein and Yang-Mills-Einstein supergravity theories in four dimensions*

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Abstract: We study unified $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories (MESGTs) and unified Yang-Mills Einstein supergravity theories (YMESGTs) in four dimensions. As their defining property, these theories admit the action of a global or local symmetry group that is (i) *simple*, and (ii) acts *irreducibly* on *all* the vector fields of the theory, including the "graviphoton". Restricting ourselves to the theories that originate from five dimensions via dimensional reduction, we find that the generic Jordan family of MESGTs with the scalar manifolds $\mathcal{M} = [\mathrm{SU}(1,1)/\mathrm{U}(1)] \times [\mathrm{SO}(2,n)/\mathrm{SO}(2) \times \mathrm{SO}(n)]$ are all unified in four dimensions with the unifying global symmetry group $\mathrm{SO}(2,n)$. Of these theories only one can be gauged so as to obtain a unified YMESGT with the gauge group $\mathrm{SO}(2,1)$. Three of the four magical supergravity theories defined by simple euclidean Jordan algebras $J_3^{\mathbb{A}}(\mathbb{A} = \mathbb{C}, \mathbb{H}, \mathbb{O})$ of degree 3 are unified MESGTs in four dimensions. The MESGTs defined by $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{O}}$ can furthermore be gauged so as to obtain a 4D unified YMESGT with gauge groups $\mathrm{SO}(3,2)$ and $\mathrm{SO}(6,2)$, respectively. The generic non-Jordan family and the theories whose scalar manifolds are homogeneous but not symmetric do not lead to unified MESGTs in four dimensions. The three infinite families of unified five-dimensional MESGTs defined by simple lorentzian Jordan algebras of degree $p \neq 4$, whose scalar manifolds are non-homogeneous, do not lead directly to unified MESGTs in four dimensions under dimensional reduction. However, since their manifolds are non-homogeneous we are not able to completely rule out the existence of symplectic sections in which these theories become unified in four dimensions.

Keywords: Extended Supersymmetry, Supersymmetry and Duality, Supergravity Models.

Contents

1. Introduction

As is well-known, starting from a few well-established principles, the no-go theorem of Coleman and Mandula [1] rules out the existence of internal "bosonic" symmetries (continuous symmetries described by Lie groups) that can interpolate between particles of different spin. As is also well-known, supersymmetry circumvents this theorem, because it is built upon "fermionic" symmetry generators with anticommuting parameters and continuous supersymmetry transformations that correspond to Lie supergroups [2, 3].

One of the early motivations to introduce the concept of space-time supersymmetry into particle physics was the hope that it might provide a link between matter particles and gauge bosons or perhaps even relate gravity with the Yang-Mills interactions of the standard model. As we now know, these ideas cannot be directly realized in the phenomenologically

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interesting $\mathcal{N}=1$ supersymmetric extensions of the standard model, because

1. no known standard model particle in these models is related by a supersymmetry transformation to another standard model particle of different spin

2. the graviton cannot be a superpartner of a standard model gauge boson in $\mathcal{N}=1$ supergravity.

In fact, as for unification and simplification, it seems that the main virtues of low energy supersymmetry are the technical resolution of the gauge hierarchy problem in grand unified theories (GUTs) and an improved convergence of the three standard model couplings at the GUT scale [4].

The second point 2, i.e., the lack of entanglement (or "unification") of Yang-Mills sectors and gravity in 4D, $\mathcal{N}=1$ supergravity, is easily traced back to the fact that both sectors live in two different types of supermultiplets and that there are no vector fields in the $\mathcal{N}=1$ supergravity multiplet. This is no longer true in *extended* supergravity theories, and, in principle, there could be a "unification" of Yang-Mills and gravitational interactions in extended supergravity due to one or two of the following (well-known) mechanisms:

1. Whereas the $\mathcal{N}=1$ supergravity multiplet contains no physical vector fields, such vector fields ("graviphotons") do occur in all supergravity multiplets with $\mathcal{N}\geq 2$. These graviphotons can often act as gauge fields for non-trivial Yang-Mills-type gauge symmetries, and if they do, supersymmetry can be viewed as a "unifying" symmetry that connects gravity and the corresponding Yang-Mills interactions. The prime example of such a theory in four dimensions is the SO(8) gauged $\mathcal{N}=8$ supergravity theory by de Wit and Nicolai [5].

2. For $\mathcal{N}\leq 4$ supersymmetry, the supergravity multiplet can also be coupled to additional vector multiplets. In such a situation, supersymmetry, just as before, directly connects the graviton to the graviphotons, but, at the linearized level, not to the vector fields from the vector multiplets. However, the non-linear couplings in all $\mathcal{N}=4$ supergravity theories and in many $\mathcal{N}=2$ theories allow for certain non-compact global symmetries that interpolate between the vector fields in the supergravity multiplet and the vector fields of the vector multiplets. In some cases, these interpolating non-compact symmetries can also be gauged. When this is the case, the noncompact symmetry generators map vector fields from vector multiplets to vector fields in the supergravity multiplet, which in turn are then linked to the graviton by supersymmetry.

In both of these setups, a particularly symmetric situation occurs when the Yang-Mills-type gauge group is (*a*) *simple* and (*b*) acts *irreducibly* on (*c*) *all* the vector fields of the theory. For theories of type 1 with only the supergravity multiplet present, the above-mentioned SO(8) gauged $\mathcal{N}=8$ supergravity (as well as its cousins with the non-compact analogues of SO(8)) form an example of this highly symmetric type of theories.

For theories of type 2, in which also vector multiplets are present, it is much harder to find examples with *simple* gauge groups that act *irreducibly* on *all* vector fields (including the "graviphotons"). Curiously, such theories do exist, however, as was first pointed out in [6] in the context of 5D, $\mathcal{N}=2$ supergravity coupled to vector multiplets. In [6, 7], such supergravity theories with vector multiplets and simple gauge groups that act irreducibly on all vector fields were called "unified Yang-Mills-Einstein supergravity theories" (unified YMESGTs), and their existence was studied in five dimensions [7]. It is easy to see that unified YMESGTs are impossible for 5D, $\mathcal{N}=4$ supergravity coupled to $\mathcal{N}=4$ vector multiplets [8], leaving $\mathcal{N}=2$ supergravity theories as the only possible candidates in five dimensions. As was shown in [7], if one restricts oneself to symmetric or homogeneous target spaces, there is precisely one 5D, $\mathcal{N}=2$ unified YMESGT, which describes the coupling of 14 vector multiplets to supergravity with the gauge group $\mathrm{SU}(3,1)$ (which is 15-dimensional and, hence also involves the vector field from the gravity multiplet) [6]. Dropping this restriction to symmetric or homogeneous scalar manifolds, however, allows for an infinite tower of unified YMESGTs, in which the target spaces are non-homogeneous and the gauge groups are of the form $\mathrm{SU}(1,N)$ for arbitrary $N\geq 2$.

Turning off the gauge coupling of a unified YMESGT gives a "unified Maxwell-Einstein supergravity theory" (unified MESGT), in which now a *global* simple symmetry group of the lagrangian acts irreducibly on all the vector fields. By contrast, not all unified MESGTs can be turned into unified YMESGTs, as not every global symmetry group can be gauged in extended supergravity. In this sense, unified YMESGTs derive, in general, only from a subset of the possible unified MESGTs. In [7], it was shown that, in five dimensions, there are precisely four unified MESGTs if one restricts oneself to symmetric or homogeneous target spaces. These four theories had been constructed in [9]. Again, relaxing the restriction to symmetric or homogeneous target spaces greatly increases the possibilities, and three infinite families of unified MESGTs as well as one exceptional one were found in [7]. Interestingly, all the known unified MESGTs and YMESGTs in five dimensions are in some way based on Jordan algebras, which hints at some deep mathematical structure that underlies these theories.

Motivated by the idea that physics may be effectively five-dimensional over some range of distance scales, possible compact gauge groups in five-dimensional, $\mathcal{N}=2$ supergravity theories with vector, tensor and hypermultiplets were studied and classified in [10]. The unified theories with noncompact gauge groups constructed in [7] extend greatly the options for noncompact gauge groups. Some interesting applications of some of the theories constructed and classified in [10] and [7] in the context of 5D GUT model building along the lines of [11] have recently been studied in [12].[1]

Motivated by possible physical applications as well as the goal of uncovering further their connection to some deep underlying mathematical structures, we extend, in this paper, the analysis of [7] to $\mathcal{N}=2$ unified MESGTs and unified YMESGTs in four dimensions. A complete classification of such theories in four dimensions is complicated by the fact that the 4D $\mathcal{U}$-duality group is in general realized only on-shell and that the prepotentials in 4D no longer have to be cubic polynomials. In this paper we will restrict our analysis to those 4D theories that can be obtained from known five-dimensional $\mathcal{N}=2$ MESGTs or

[1] See also [13, 14]

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YMESGTs by dimensional reduction. The theories that have a five-dimensional origin are physically interesting for a number of reasons. They are relevant for describing the classical large volume limits of IIA string theory compactifications on Calabi-Yau manifolds (or for M-theory compactifications on certain singular Calabi-Yau manifolds times S^1 that lead to YMESGTs coupled to hypermultiplets [15 – 19]). Therefore, these theories are important for finding the M-theoretic origins of orbifold-GUTs. Furthermore the four-dimensional $\mathcal{N} = 2$ no-scale supergravity models have their origins in five dimensions [20 – 22].

The organization of this paper is as follows. In sections 2 and 3 we review, respectively, the $\mathcal{N} = 2$ MESGTs and the known classification of unified 5D MESGTs and YMESGTs. In section 4 we recall the bosonic lagrangian of the $\mathcal{N} = 2$ MESGTs dimensionally reduced to four dimensions and discuss the very special geometries of these theories. Our main results are in section 5 where we give a classification of the 4D unified MESGTs and unified YMESGTs that originate from five dimensions. The conclusion section 6 summarizes our main results. Appendix A gives a review of basic facts about Jordan algebras and their symmetries that are relevant to our work. Appendix B gives a brief review of the symplectic formulation of four-dimensional $\mathcal{N} = 2$ MESGTs.

2. 5D, $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories

In this section we briefly summarize the structure of general 5D, $\mathcal{N} = 2$ Maxwell-Einstein supergravity theories (MESGTs) [9].[2]

A 5D, $\mathcal{N} = 2$ MESGT describes the coupling of $\tilde{n}$ vector multiplets to minimal, $\mathcal{N} = 2$ supergravity in five dimensions. The field content of the supergravity multiplet is given by the fünfbein e_μ^m, two gravitini Ψ_μ^i $(i = 1, 2)$ and one vector field A_μ (the graviphoton). An $\mathcal{N} = 2$ vector multiplet contains a vector field A_μ, two spin-1/2 fermions λ^i and one real scalar field φ. The fermions of each of these multiplets transform as doublets under the $\mathrm{USp}(2)_R \cong \mathrm{SU}(2)_R$ R-symmetry group of the $\mathcal{N} = 2$ Poincaré superalgebra; all other fields are $\mathrm{SU}(2)_R$-inert.

Putting everything together, the total field content of an $\mathcal{N} = 2$ MESGT in 5D is thus

$$\left\{ e_\mu^m, \Psi_\mu^i, A_\mu^{\tilde{I}}, \lambda^{i\tilde{a}}, \varphi^{\tilde{x}} \right\} \tag{2.1}$$

with

$$\begin{aligned} \tilde{I} &= 0, 1, \ldots, \tilde{n} \\ \tilde{a} &= 1, \ldots, \tilde{n} \\ \tilde{x} &= 1, \ldots, \tilde{n}\,. \end{aligned}$$

As usual, we have combined the graviphoton with the $\tilde{n}$ vector fields of the $\tilde{n}$ vector multiplets into a single $(\tilde{n}+1)$-plet of vector fields $A_\mu^{\tilde{I}}$ labelled by the index $\tilde{I}$. The indices $\tilde{a}, \tilde{b}, \ldots$ and $\tilde{x}, \tilde{y}, \ldots$ denote the flat and curved indices, respectively, of the $\tilde{n}$-dimensional target manifold, $\mathcal{M}$, of the scalar fields.

[2]Our conventions coincide with those of ref. [9, 6, 23]. In particular, we will use the mostly positive metric signature $(- + + + +)$ and impose the 'symplectic' Majorana condition on all fermionic quantities.

The bosonic part of the lagrangian is given by (for the fermionic part and further details we refer to [9])

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bosonic}} = & -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}} F^{\tilde{I}}_{\mu\nu} F^{\tilde{J}\mu\nu} - \frac{1}{2} g_{\tilde{x}\tilde{y}} (\partial_\mu \varphi^{\tilde{x}})(\partial^\mu \varphi^{\tilde{y}}) + \\ & + \frac{e^{-1}}{6\sqrt{6}} C_{\tilde{I}\tilde{J}\tilde{K}} \varepsilon^{\mu\nu\rho\sigma\lambda} F^{\tilde{I}}_{\mu\nu} F^{\tilde{J}}_{\rho\sigma} A^{\tilde{K}}_{\lambda} \,, \end{aligned} \tag{2.2}$$

where e and R, respectively, denote the fünfbein determinant and scalar curvature of spacetime. $F^{\tilde{I}}_{\mu\nu}$ are the abelian field strengths of the vector fields $A^{\tilde{I}}_{\mu}$. The metric, $g_{\tilde{x}\tilde{y}}$, of the scalar manifold $\mathcal{M}$ and the matrix $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ both depend on the scalar fields $\varphi^{\tilde{x}}$. The completely symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$, by contrast, is constant.

As was observed in reference [9], the entire $\mathcal{N} = 2$ MESGT (including the fermionic terms and the supersymmetry transformation laws that we have suppressed) is uniquely determined by $C_{\tilde{I}\tilde{J}\tilde{K}}$ [9]. More explicitly, $C_{\tilde{I}\tilde{J}\tilde{K}}$ defines a cubic polynomial, $\mathcal{V}(h)$, in $(\tilde{n}+1)$ real variables $h^{\tilde{I}}$ $(\tilde{I} = 0, 1, \ldots, \tilde{n})$,

$$\mathcal{V}(h) := C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} \,. \tag{2.3}$$

This polynomial defines a metric, $a_{\tilde{I}\tilde{J}}$, in the (auxiliary) space $\mathbb{R}^{(\tilde{n}+1)}$ spanned by the $h^{\tilde{I}}$:

$$a_{\tilde{I}\tilde{J}}(h) := -\frac{1}{3} \frac{\partial}{\partial h^{\tilde{I}}} \frac{\partial}{\partial h^{\tilde{J}}} \ln \mathcal{V}(h) \,. \tag{2.4}$$

The $\tilde{n}$-dimensional target space, $\mathcal{M}$, of the scalar fields $\varphi^{\tilde{x}}$ can then be represented as the hypersurface [9]

$$\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1 \,, \tag{2.5}$$

with $g_{\tilde{x}\tilde{y}}$ being the pull-back of (2.4) to $\mathcal{M}$:

$$g_{xy}(\varphi) = \frac{3}{2} (\partial_x h^{\tilde{I}})(\partial_y h^{\tilde{J}}) a_{\tilde{I}\tilde{J}}|_{\mathcal{V}=1} \,. \tag{2.6}$$

Finally, the quantity $\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi)$ appearing in (2.2) is given by the componentwise restriction of $a_{\tilde{I}\tilde{J}}$ to $\mathcal{M}$:

$$\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi) = a_{\tilde{I}\tilde{J}}|_{\mathcal{V}=1} \,.$$

The physical requirement of the positivity of the kinetic energy terms for the scalar and vector fields requires $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ to be positive definite. This requirement induces constraints on the possible $C_{\tilde{I}\tilde{J}\tilde{K}}$, and in [9] it was shown that any $C_{\tilde{I}\tilde{J}\tilde{K}}$ that satisfy these constraints can be brought, by a linear transformation, to the following form

$$C_{000} = 1 \,, \qquad C_{0ij} = -\frac{1}{2}\delta_{ij} \,, \qquad C_{00i} = 0 \,, \tag{2.7}$$

with the remaining coefficients C_{ijk} $(i, j, k = 1, 2, \ldots, \tilde{n})$ being completely arbitrary. Such a basis is called a "canonical basis". The arbitrariness of the C_{ijk} in a canonical basis implies that, for a fixed number $\tilde{n}$ of vector multiplets, different target manifolds $\mathcal{M}$ are, in general, possible.

3. Unified MESGTs in five dimensions

In the previous section, we have treated the vector fields from the vector multiplets and the one that stems from the supergravity multiplet on an equal footing. *A priori*, this is of course just a formal procedure that nicely displays the underlying very special geometry of the theory. As was pointed out already in [9], however, for some very special tensors $C_{\tilde{I}\tilde{J}\tilde{K}}$, the corresponding MESGTs can have surprisingly large symmetry groups, G, that are

(i) *simple* and

(ii) act *irreducibly* on *all* the vector fields, including the "graviphoton".[3]

As in [7], we refer to such theories as "unified MESGTs". More precisely, a unified MESGT is a MESGT in which the coefficients $C_{\tilde{I}\tilde{J}\tilde{K}}$ form an invariant third rank tensor of a *simple* group G that acts *irreducibly* on both the vector fields $A_\mu^{\tilde{I}}$ and the embedding functions $h^{\tilde{I}}$ of the very special geometry:

$$h^{\tilde{I}} \to B^{\tilde{I}}{}_{\tilde{J}} h^{\tilde{J}} , \qquad A_\mu^{\tilde{I}} \to B^{\tilde{I}}{}_{\tilde{J}} A_\mu^{\tilde{J}} \tag{3.1}$$

with

$$B^{\tilde{I}'}{}_{\tilde{I}} B^{\tilde{J}'}{}_{\tilde{J}} B^{\tilde{K}'}{}_{\tilde{K}} C_{\tilde{I}'\tilde{J}'\tilde{K}'} = C_{\tilde{I}\tilde{J}\tilde{K}} \, .$$

As the entire lagrangian (2.2) is uniquely determined by the $C_{\tilde{I}\tilde{J}\tilde{K}}$, any invariance of the $C_{\tilde{I}\tilde{J}\tilde{K}}$ is automatically a symmetry of the whole lagrangian.[4] On the scalar manifold $\mathcal{M}$, such symmetry transformations act as isometries, which becomes evident if one rewrites the kinetic term for the scalar fields as [9, 24]

$$-\frac{1}{2} g_{\tilde{x}\tilde{y}} (\partial_\mu \varphi^{\tilde{x}})(\partial^\mu \varphi^{\tilde{y}}) = \frac{3}{2} C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} \partial_\mu h^{\tilde{J}} \partial^\mu h^{\tilde{K}} \, ,$$

with the $h^{\tilde{I}}$ being constrained according to (2.5). Hence, finding a unified MESGT is equivalent to finding an irreducible representation of a simple group G with an invariant symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ of rank three that gives rise to positive definite metrics g_{xy} and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ (i.e., that can be brought to the canonical form (2.7)).

Before the work of reference [7], only four unified MESGTs in five dimensions were known. These are based on scalar manifolds that are symmetric spaces :

$$\begin{aligned}
\mathcal{M} &= \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) && (\tilde{n} = 5) \\
\mathcal{M} &= \mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3) && (\tilde{n} = 8) \\
\mathcal{M} &= \mathrm{SU}^*(6)/\,\mathrm{Usp}(6) && (\tilde{n} = 14) \\
\mathcal{M} &= E_{6(-26)}/F_4 && (\tilde{n} = 26),
\end{aligned} \tag{3.2}$$

[3] It should be noted that it is the G-invariant linear combination $h_{\tilde{I}} F_{\mu\nu}^{\tilde{I}}$ that appears in the gravitino transformation law [9]. In a given spacetime background the $h_{\tilde{I}}$ are constant, in which case the combination $h_{\tilde{I}} A_\mu^{\tilde{I}}$ might be called the "graviphoton". Although it is not important for this paper, one can think of it as the vector field $A_\mu^{\tilde{I}} c_{\tilde{I}}$ where $c_{\tilde{I}}$ is the basepoint at which $\overset{\circ}{a}_{\tilde{I}\tilde{J}}|_c = \delta_{\tilde{I}\tilde{J}}$ (see [9]).

[4] This statement is of course true for all symmetries of the $C_{\tilde{I}\tilde{J}\tilde{K}}$, independently of whether G is simple or acts irreducibly on the $A_\mu^{\tilde{I}}$ and the $h^{\tilde{I}}$.

where we have indicated the number of vector multiplets, $\tilde{n}$, for each of these theories. In these cases, the symmetry groups G of these theories are simply the isometry groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SU}^*(6)$ and $E_{6(-26)}$, respectively, under which the, respectively, 6, 9, 15 and 27 vector fields $A_\mu^{\tilde{I}}$ transform irreducibly [9]. Thus, according to our definition, all of these four theories are unified MESGTs.

The fact that these theories are based on symmetric target spaces certainly makes them a bit special. However, being a symmetric space and giving rise to a unified symmetry is by no means the same thing. For one thing, all the other symmetric spaces that are allowed in 5D, $\mathcal{N}=2$ MESGTs do *not* give rise to unified MESGTs. Indeed, it is easy to verify that the two other existing series of symmetric target spaces [9, 25],

$$\mathcal{M} = \frac{\mathrm{SO}(\tilde{n}-1,1)}{\mathrm{SO}(\tilde{n}-1)} \times \mathrm{SO}(1,1)\,, \qquad \tilde{n} \geq 1\,, \tag{3.3}$$

and

$$\mathcal{M} = \frac{\mathrm{SO}(1,\tilde{n})}{\mathrm{SO}(\tilde{n})}\,, \qquad \tilde{n} > 1\,, \tag{3.4}$$

do not give rise to any simple symmetry group that acts irreducibly on all the vector fields [9, 7]. Thus, among the theories based on symmetric spaces, only four are unified MESGTs (3.2).

Moreover, in [7], it was shown that there can also be unified MESGTs whose target manifolds are *not* symmetric spaces. In fact, these form the large majority of unified MESGTs, as there are infinitely many of them. Interestingly, these novel unified MESGTs do have something in common with the theories of the symmetric families (3.2) and (3.3): They are all related to Jordan algebras, albeit in a different way than the symmetric space theories (3.2) and (3.3).[5]

In order to see this, let us recall the definition of a Jordan algebra (see also appendix A):

Definition 1. *A Jordan algebra over a field $\mathbb{F}$ (which we take to be $\mathbb{R}$ or $\mathbb{C}$) is an algebra, J, over $\mathbb{F}$ with a symmetric product, $\circ$,*

$$X \circ Y = Y \circ X \in J\,, \qquad \forall\, X, Y \in J\,, \tag{3.5}$$

that satisfies the Jordan identity

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2\,, \tag{3.6}$$

where $X^2 \equiv (X \circ X)$.

The Jordan identity (3.6) is automatically satisfied when the product $\circ$ is associative, but (3.6) does not imply associativity. In other words, a Jordan algebra is commutative, but in general not associative. It is easy to verify that the hermitean operators acting in a Hilbert space form a Jordan algebra with the product "$\circ$" being one half the anticommutator.[6] In fact, historically, Jordan algebras were introduced in an attempt to generalize the

[5] The third family of symmetric spaces (3.4), is not related to Jordan algebras.

[6] The prefactor one half in front of the anticommutator allows one to represent the identity element by a unit matrix for finite dimensional Jordan algebras. For the importance of the prefactor one half for different formulations of Jordan algebras see the book by McCrimmon [29].

formalism of quantum mechanics by capturing the algebraic essence of hermitean operators corresponding to observables without reference to the underlying Hilbert space on which they act [26].

A particular subclass of Jordan algebras is formed by the *euclidean* Jordan algebras:

Definition 2. *A* euclidean *(or* formally real*) Jordan algebra is a Jordan algebra for which the condition* $X \circ X + Y \circ Y = 0$ *implies that* $X = Y = 0$ *for all* $X, Y \in J$*. The automorphism groups of euclidean Jordan algebras are always compact, which suggests the alternate name "compact Jordan algebras" as is sometimes used in the literature.*

For every Jordan algebra J, one can define a norm form, $N : J \to \mathbb{R}$, that satisfies the composition property [27]

$$N[2X \circ (Y \circ X) - (X \circ X) \circ Y] = N^2(X) N(Y) . \tag{3.7}$$

The degree, p, of the norm form is defined by $N(\lambda X) = \lambda^p N(X)$, where $\lambda \in \mathbb{R}$; p is also called the degree of the Jordan algebra.

It was shown in [9] that the norm form, N, of a euclidean Jordan algebra of degree three ($p = 3$) can be identified with the cubic polynomial, $\mathcal{V}$, of a physically sensible MESGT. More precisely, the cubic polynomials of euclidean Jordan algebras of degree three satisfy all the constraints required by supersymmetry and the positivity of kinetic terms. In addition, these polynomials satisfy another relation, called the adjoint identity, which implies that the corresponding scalar manifolds are symmetric spaces of the form

$$\mathcal{M} = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)} ,$$

where $\mathrm{Str}_0(J)$ and $\mathrm{Aut}(J)$ denote, respectively, the reduced structure and automorphism groups of the underlying Jordan algebra[7] [9, 28]. The MESGTs whose cubic polynomials arise in this way, are precisely the first two families (3.2) and (3.3). The relevant Jordan algebras are

1. The "magical Jordan family" (3.2) [8] corresponds to the four *simple* euclidean Jordan algebras of degree 3. These simple Jordan algebras are denoted by $J_3^{\mathbb{R}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{O}}$ and are isomorphic to the hermitean (3×3)-matrices over the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ with the product being one half the anticommutator:

$$\begin{aligned}
J_3^{\mathbb{R}} &: \quad \mathcal{M} = \mathrm{SL}(3,\mathbb{R}) / \mathrm{SO}(3) && (\tilde{n} = 5) \\
J_3^{\mathbb{C}} &: \quad \mathcal{M} = \mathrm{SL}(3,\mathbb{C}) / \mathrm{SU}(3) && (\tilde{n} = 8) \\
J_3^{\mathbb{H}} &: \quad \mathcal{M} = \mathrm{SU}^*(6) / \mathrm{USp}(6) && (\tilde{n} = 14) \\
J_3^{\mathbb{O}} &: \quad \mathcal{M} = E_{6(-26)} / F_4 && (\tilde{n} = 26) .
\end{aligned} \tag{3.8}$$

[7] The reduced structure group $\mathrm{Str}_0(J)$ is simply the invariance group of the norm form, N, of the corresponding Jordan algebra J. As such, it is, for the above Jordan algebras, isomorphic to the symmetry group G of the corresponding MESGTs.

[8] The name "magical" derives from the deep connection with the "magic square" of Freudenthal, Rosenfeld and Tits [30].

The cubic norm form, N, of these Jordan algebras is given by the determinant of the corresponding hermitean (3×3)-matrices.

2. For the "generic Jordan family" (3.3), the scalar manifolds factorize:

$$\mathcal{M} = \frac{\mathrm{SO}(\tilde{n}-1,1)}{\mathrm{SO}(\tilde{n}-1)} \times \mathrm{SO}(1,1)\,.$$

This is also reflected in the underlying Jordan algebras, which are now reducible and of the form $J = \mathbb{R} \oplus \Sigma_{\tilde{n}}$. Here, $\Sigma_{\tilde{n}}$ is a Jordan algebra of degree $p = 2$ associated with a quadratic norm form in $\tilde{n}$ dimensions that has a "minkowskian signature" $(+,-,\ldots,-)$. A simple realization of $\Sigma_{\tilde{n}}$ is provided by $(\tilde{n}-1)$ Dirac gamma matrices γ^i , where $i,j,\ldots = 1,\ldots,(\tilde{n}-1)$ of an $(\tilde{n}-1)$ dimensional euclidean space together with the identity matrix $\gamma^0 = \mathbf{1}$, and the Jordan product $\circ$ being one half the anticommutator:

$$\begin{aligned}
\gamma^i \circ \gamma^j &= \frac{1}{2}\{\gamma^i,\gamma^j\} = \delta^{ij}\mathbf{1} \\
\gamma^0 \circ \gamma^0 &= \frac{1}{2}\{\gamma^0,\gamma^0\} = \mathbf{1} \\
\gamma^i \circ \gamma^0 &= \frac{1}{2}\{\gamma^i,\gamma^0\} = \gamma^i\,.
\end{aligned} \tag{3.9}$$

The norm of a general element $X = X_0\gamma^0 + X_i\gamma^i$ of $\Sigma_{\tilde{n}}$ is defined as

$$N(X) = \frac{1}{2^{[\tilde{n}/2]}}\,\mathrm{Tr}\,X\bar{X} = X_0X_0 - X_iX_i\,,$$

where

$$\bar{X} \equiv X_0\gamma^0 - X_i\gamma^i\,.$$

The norm of a general element $y \oplus X$ of the non-simple Jordan algebra $J = \mathbb{R} \oplus \Sigma_{\tilde{n}}$ is simply given by $yN(X)$.

Thus, identifying the cubic polynomial $\mathcal{V}$ defined by the supergravity coefficients $C_{\tilde{I}\tilde{J}\tilde{K}}$ with the norm form, N, of a euclidean Jordan algebra of degree three yields precisely the MESGTs based on the scalar manifolds (3.2) and (3.3).

Let us now describe how the novel unified MESGTs found in [7] are related to Jordan algebras. In those theories, one does not relate the $C_{\tilde{I}\tilde{J}\tilde{K}}$ to the *norm form*, N, of a *euclidean* Jordan algebra of degree three, but instead one relates them to the *structure constants* of a *lorentzian* Jordan algebra.[9]

Let us make this more precise. We first note that all the Jordan algebras of $(n \times n)$ hermitean matrices over various division algebras are euclidean (compact) Jordan algebras, which implies that their automorphism groups are compact. In particular, the Jordan algebras of degree 3 discussed in 1. and 2. are all euclidean. The three Jordan algebras $J_3^{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}}$ described in 1. are just special cases of the Jordan algebras of the $(n\times n)$ hermitean

[9] In [7] the lorentzian Jordan algebras were called minkowskian Jordan algebras.

matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.[10] These higher-dimensional analogues are also all euclidean Jordan algebras with compact automorphism groups. Their norm forms, N, are again given by the determinants of the matrices, which implies that the degree of $J_n^{\mathbb{A}}$ is n, singling out $J_3^{\mathbb{A}}$ as the only members of these families with cubic norms.

Now, the euclidean Jordan algebras $J_n^{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}}$ and $J_3^{\mathbb{O}}$ also have non-euclidean analogues $J_{(q,n-q)}^{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}}$ (again with $n \geq 3$) and $J_{(1,2)}^{\mathbb{O}}$, respectively. These non-euclidean Jordan algebras are realized by matrices that are hermitean with respect to a non-euclidean "metric" η with signature $(q, n-q)$:

$$(\eta X)^{\dagger} = \eta X \qquad \forall X \in J_{(q,n-q)}^{\mathbb{A}} \,. \tag{3.10}$$

Obviously, if we choose η to have a euclidean signature, we obtain back the euclidean (compact) Jordan algebras.

Consider now lorentzian Jordan algebras $J_{(1,N)}^{\mathbb{A}}$ with degree $p = N+1$ defined by choosing η to be the lorentzian (or "minkowskian") metric $\eta = (-,+,+,\ldots,+)$. A general element, U, of $J_{(1,N)}^{\mathbb{A}}$ can be written in the form

$$U = \begin{pmatrix} x & -Y^{\dagger} \\ Y & Z \end{pmatrix}, \tag{3.11}$$

where Z is an element of the euclidean subalgebra $J_N^{\mathbb{A}}$ (i.e., it is a hermitean $(N \times N)$-matrix over $\mathbb{A}$), $x \in \mathbb{R}$, and Y denotes an N-dimensional column vector over $\mathbb{A}$. Under their (non-compact) automorphism group, $\mathrm{Aut}(J_{(1,N)}^{\mathbb{A}})$, these simple Jordan algebras $J_{(1,N)}^{\mathbb{A}}$ decompose into an irreducible representation formed by the traceless elements plus a singlet, which is given by the identity element of $J_{(1,N)}^{\mathbb{A}}$ (i.e., by the unit matrix $U = \mathbf{1}$):

$$J_{(1,N)}^{\mathbb{A}} = \mathbf{1} \oplus \{\text{traceless elements}\} \equiv \mathbf{1} \oplus J_{(1,N)_0}^{\mathbb{A}} \,. \tag{3.12}$$

The traceless elements do not close under the Jordan product, $\circ$, but one can define a symmetric product, $\star$, under which the traceless elements do close as follows:[11]

$$A \star B := A \circ B - \frac{1}{(N+1)} \mathrm{tr}(A \circ B) \mathbf{1}$$

where $A, B \in J_{(1,N)_0}^{\mathbb{A}}$ and $\circ$ is the Jordan product

$$A \circ B = \frac{1}{2}(AB + BA) \,.$$

Thus, the structure constants (d-symbols) of the traceless elements under the symmetric $\star$ product will be invariant tensors of the automorphism groups $\mathrm{Aut}(J_{(1,N)}^{\mathbb{A}})$ of the Jordan

[10] Hermitean $(n \times n)$ matrices over the octonions do not form Jordan algebras for $n > 3$.

[11] Such a product was introduced among the hermitean generators of $\mathrm{SU}(N)$ by Michel and Radicati some time ago [31]. Note that hermitean generators of $\mathrm{SU}(N)$ are in one-to-one correspondence with the traceless elements of $J_N^{\mathbb{C}}$. The "symmetric" algebras with the star product $\star$ do not have an identity element.

algebras. Denoting the traceless elements as $T_{\tilde{I}}$ ($\tilde{I} = 0, \ldots, (D-2)$) with D being the dimension of $J^{\mathbb{A}}_{(1,N)}$, we have

$$T_{\tilde{I}} \star T_{\tilde{J}} = d_{\tilde{I}\tilde{J}}{}^{\tilde{K}} T_{\tilde{K}} \,.$$

The d-symbols are then given by

$$d_{\tilde{I}\tilde{J}\tilde{K}} \equiv d_{\tilde{J}\tilde{K}}{}^{\tilde{L}} \tau_{\tilde{L}\tilde{I}} = \frac{1}{2}\mathrm{tr}(T_{\tilde{I}}\{T_{\tilde{J}}, T_{\tilde{K}}\}) = \mathrm{tr}(T_{\tilde{I}} \circ (T_{\tilde{J}} \circ T_{\tilde{K}}))\,, \tag{3.13}$$

where [12]

$$\tau_{\tilde{L}\tilde{I}} = \mathrm{tr}(T_{\tilde{L}} \circ T_{\tilde{I}})\,.$$

The $d_{\tilde{I}\tilde{J}\tilde{K}}$ are completely symmetric in their indices, and as $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$ acts irreducibly on the traceless elements $T_{\tilde{I}}$, the $d_{\tilde{I}\tilde{J}\tilde{K}}$ are a promising candidate for the $C_{\tilde{I}\tilde{J}\tilde{K}}$ of a unified MESGT. What remains to check, however, is whether the metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ on the resulting scalar manifold $\mathcal{M}$ are really positive definite. It was shown in [7] that this is true if and only if the signature of η is really $(-,+,\ldots,+)$.

Thus, putting everything together, if one identifies the d-symbols (3.13) of the traceless elements of a lorentzian Jordan algebra $J^{\mathbb{A}}_{(1,N)}$ with the $C_{\tilde{I}\tilde{J}\tilde{K}}$ of a MESGT: $C_{\tilde{I}\tilde{J}\tilde{K}} = d_{\tilde{I}\tilde{J}\tilde{K}}$, one obtains a unified MESGT, in which all the vector fields transform irreducibly under the simple automorphism group $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$ of that Jordan algebra.

For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ one obtains in this way three infinite families of physically acceptable unified MESGTs (one for each $N \geq 2$). For the octonionic case, the situation is a bit different. The d-symbols of the octonionic lorentzian hermitean $(N+1)\times(N+1)$-matrices with the anticommutator product all lead to positive definite metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$, i.e., to physically acceptable MESGTs. For $N \neq 2$, however, these octonionic hermitean matrix algebras are no longer Jordan algebras. Surprisingly, the automorphism groups of these octonionic algebras for $N \geq 3$ do not contain the automorphism group $F_{4(-20)}$ of $J^{\mathbb{O}}_{(1,2)}$ as a subgroup [32]. Instead, the automorphism groups for $N \geq 3$ are direct product groups of the form $\mathrm{SO}(N,1) \times G_2$. None of these two factors acts irreducibly on all the traceless elements, and hence the corresponding $\mathcal{N} = 2$ MESGTs are not unified theories. Thus, the $\mathcal{N} = 2$ MESGT defined by the exceptional lorentzian Jordan algebra $J^{\mathbb{O}}_{(1,2)}$ is the only *unified* MESGT of this infinite tower of otherwise acceptable octonionic theories.

All these results are summarized in table 1, which lists all the simple lorentzian Jordan algebras of type $J^{\mathbb{A}}_{(1,N)}$, their automorphism groups, and the numbers of vector and scalar fields in the unified MESGTs defined by them.

Note that the number of vector fields for the theories defined by $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ are 9, 15 and 27, respectively. These are exactly the same numbers of vector fields in the magical theories based on the norm forms of the euclidean Jordan algebras $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$, respectively. As was shown in [7], this is not an accident; the magical MESGTs based on $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$ found in [9] are *equivalent* (i.e. the cubic polynomials $\mathcal{V}(h)$ agree) to the ones defined by the lorentzian algebras $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$, respectively. This is a

[12] One can choose the elements $T_{\tilde{I}}$ such that $\tau_{\tilde{I}\tilde{J}} = \delta_{\tilde{I}\tilde{J}}$ ($\tau_{\tilde{I}\tilde{J}} = -\delta_{\tilde{I}\tilde{J}}$) for two compact (noncompact) elements $T_{\tilde{I}}$ and $T_{\tilde{J}}$ and zero otherwise.

J	D	$\mathrm{Aut}(J)$	No. of vector fields	No. of scalars
$J^{\mathbb{R}}_{(1,N)}$	$\frac{1}{2}(N+1)(N+2)$	$\mathrm{SO}(N,1)$	$\frac{1}{2}N(N+3)$	$\frac{1}{2}N(N+3)-1$
$J^{\mathbb{C}}_{(1,N)}$	$(N+1)^2$	$\mathrm{SU}(N,1)$	$N(N+2)$	$N(N+2)-1$
$J^{\mathbb{H}}_{(1,N)}$	$(N+1)(2N+1)$	$\mathrm{USp}(2N,2)$	$N(2N+3)$	$N(2N+3)-1$
$J^{\mathbb{O}}_{(1,2)}$	27	$F_{4(-20)}$	26	25

Table 1: List of the simple lorentzian Jordan algebras of type $J^{\mathbb{A}}_{(1,N)}$. The columns show, respectively, their dimensions D, their automorphism groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})$, the number of vector fields $(\tilde{n}+1) = (D-1)$ and the number of scalars $\tilde{n} = (D-2)$ in the corresponding MESGTs.

consequence of the fact that the generic norms of the degree 3 simple euclidean Jordan algebras $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ coincide with the cubic norms defined over the traceless elements of degree four simple Lorentzian Jordan algebras over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ [7, 33]. This implies that the only known unified MESGT that is not covered by the above table is the magical theory of [9] based on the euclidean Jordan algebra $J_3^{\mathbb{R}}$ with $(\tilde{n}+1) = 6$ vector fields and the target space $\mathcal{M} = \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$.

Of these unified MESGTs only the family defined by $J^{\mathbb{C}}_{(1,N)}$ can be gauged so as to obtain an infinite family of unified YMESGTs with the gauge groups $\mathrm{SU}(N,1)$ [7].

Except for the theories defined by $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ the scalar manifolds of MESGTs defined by other simple lorentzian Jordan algebras are not homogeneous. Let us therefore close this section with a short description of their geometry.

To this end, one recalls that the scalar manifolds $\mathcal{M}$ of lorentzian Jordan algebraic theories in five dimensions are cubic surfaces defined by the condition

$$\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1\,,$$

where $C_{\tilde{I}\tilde{J}\tilde{K}}$ are the structure constants $C_{\tilde{I}\tilde{J}\tilde{K}} \equiv d_{\tilde{I}\tilde{J}\tilde{K}}$ of the traceless elements of the lorentzian Jordan algebra $J^{\mathbb{A}}_{(1,N)}$.

Let us now show that the scalar manifolds of 5D unified MESGTs defined by lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ for $N \geq 2$ can be embedded in the coset spaces

$$\mathcal{R} = \frac{\mathrm{Str}_0(J^{\mathbb{A}}_{(1,N)})}{\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})}\,,$$

where, just as above, $\mathrm{Str}_0(J)$ and $\mathrm{Aut}(J)$ denote, respectively, the reduced structure and automorphism groups of the Jordan algebra J. First the coset space $\mathcal{R}$ itself can be realized as a hypersurface in an $(\tilde{n}+2)$-dimensional ambient space with coordinates ξ^A $(A, B = 0, 1, \ldots, \tilde{n}+1)$ and metric

$$G_{AB}(\xi) := -\frac{1}{3}\frac{\partial}{\partial \xi^A}\frac{\partial}{\partial \xi^B}\ln \mathcal{P}(\xi)\,, \tag{3.14}$$

where

$$\mathcal{P}(\xi) = C_{AB\ldots D}\,\xi^A \xi^B \cdots \xi^D$$

is the generic norm form of the $(\tilde{n}+2)$-dimensional Jordan algebra $J^{\mathbb{A}}_{(1,N)}$ defined by a completely symmetric tensor $C_{AB\ldots D}$ of rank $N+1$.

The $(\tilde{n}+1)$-dimensional hypersurface, $\mathcal{R}$, is defined by the condition

$$\mathcal{P} = C_{AB\ldots D}\, \xi^A \xi^B \cdots \xi^D = 1\,. \tag{3.15}$$

Since the norm forms of the Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ are determinental forms, the norm $\mathcal{P}$ of an element J can be expanded in terms of the products of traces, $\mathrm{Tr}(J^m)$ of the powers of J $(m = 1, 2, \ldots, N+1)$. A general element, J, of $J^{\mathbb{A}}_{(1,N)}$ can be decomposed as

$$J = \xi^{\tilde{n}+1} 1 + e_{\tilde{I}} h^{\tilde{I}}$$

where 1 is the identity element and $e_{\tilde{I}}$ (with $\tilde{I} = 0, \ldots \tilde{n}+1$) form a basis of traceless elements of $J^{\mathbb{A}}_{(1,N)}$.[13]. Now the norm $\mathcal{P}$ depends on one more parameter, namely $\xi^{\tilde{n}+1}$, than the cubic form $\mathcal{V}$ and hence one can simultaneously impose the conditions

$$\mathcal{P} = 1 \qquad \text{and} \qquad \mathcal{V} = 1\,,$$

showing that the scalar manifold of a unified MESGT defined by a lorentzian Jordan algebra can be mapped into a cubic hypersurface in the coset space

$$\mathcal{R} = \frac{\mathrm{Str}_0(J^{\mathbb{A}}_{(1,N)})}{\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})}\,.$$

We list the coset spaces $\mathcal{R}$ defined by lorentzian Jordan algebras below:

$$\begin{aligned}
\mathcal{R}(J^{\mathbb{R}}_{(1,N)}) &= \frac{\mathrm{SL}(N+1,\mathbb{R})}{\mathrm{SO}(N,1)}\\
\mathcal{R}(J^{\mathbb{C}}_{(1,N)}) &= \frac{\mathrm{SL}(N+1,\mathbb{C})}{\mathrm{SU}(N,1)}\\
\mathcal{R}(J^{\mathbb{H}}_{(1,N)}) &= \frac{\mathrm{SU}^*(2N+2)}{\mathrm{USp}(2N,2)}\,.
\end{aligned}$$

4. Dimensional reduction of 5D, $\mathcal{N}=2$ MESGTs to four dimensions

In the previous section we have listed the known unified MESGTs in five dimensions, as given in [7]. Interestingly, all of these theories are related to Jordan algebras, but only four of them are based on symmetric target spaces. It is the purpose of this paper, to study unified MESGTs (and YMESGTs) in four space-time dimensions. A complete classification of these theories would be quite a complicated problem due to two novel features of MESGTs in four dimensions:

(i) The prepotentials of the special geometry are, in general, no longer purely cubic polynomials.

[13]In terms of the coordinates $\xi^A (A = 0, 1, \ldots, \tilde{n}+1)$ we are identifying $\xi^{\tilde{I}}$ with $h^{\tilde{I}}$ $(\tilde{I} = 0, 1, \ldots, \tilde{n})$.

(ii) They are subject to symplectic reparametrizations that may lead to lagrangians with different symmetries.

In order to cope with the first difficulty, we will restrict ourselves to those 4D theories that can be obtained from five dimensions by dimensional reduction. Some of the physical motivations for this were given above. The resulting "very special" Kähler manifolds in 4D are then based on cubic prepotentials.

In this section, we will briefly review the dimensional reduction of the 5D, $\mathcal{N} = 2$ MESGTs to four dimensions. As we did in the above section we shall restrict ourselves to the purely bosonic sector of these theories. We should stress again that by far not all four-dimensional MESGTs can be obtained by dimensional reduction from five dimensions. The metric of the target space of the four-dimensional scalar fields of dimensionally reduced theories were first obtained in [9] where a complete list of the symmetric target manifolds of theories defined by Jordan algebras were also given. These manifolds as well as those of the generic non-Jordan family [25] were later studied in [24, 34], where it was pointed out that not all the isometries of the scalar manifolds of the generic non-Jordan family are symmetries of the lagrangian.

As in [9], we choose the parametrization for the fünfbein to be[14]

$$\hat{e}_{\hat{\mu}}^{\hat{m}} = \begin{pmatrix} e^{-\frac{\sigma}{2}} e_{\mu}^{m} & 2e^{\sigma} W_{\mu} \\ 0 & e^{\sigma} \end{pmatrix}, \tag{4.1}$$

and decompose the vector fields $\hat{A}_{\hat{\mu}}^{\tilde{I}}$ into a 4D vector field, $A_{\mu}^{\tilde{I}}$, and a 4D scalar, $A^{\tilde{I}}$,

$$\hat{A}_{\hat{\mu}}^{\tilde{I}} = \begin{pmatrix} \hat{A}_{\mu}^{\tilde{I}} \\ \hat{A}_{5}^{\tilde{I}} \end{pmatrix} = \begin{pmatrix} A_{\mu}^{\tilde{I}} + 2W_{\mu} A^{\tilde{I}} \\ A^{\tilde{I}} \end{pmatrix}. \tag{4.2}$$

Using this decomposition, the dimensionally reduced lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L} = & -\frac{1}{2}R - \frac{3}{4}\partial_{\mu}\sigma\partial^{\mu}\sigma - \frac{1}{2}e^{-2\sigma}\, \overset{\circ}{a}_{\tilde{I}\tilde{J}}\, \partial_{\mu}A^{\tilde{I}}\partial^{\mu}A^{\tilde{J}} - \frac{1}{2}g_{\tilde{x}\tilde{y}}\partial_{\mu}\phi^{\tilde{x}}\partial^{\mu}\phi^{\tilde{y}} \\ & - \left(\frac{1}{2}e^{3\sigma} + e^{\sigma}\, \overset{\circ}{a}_{\tilde{I}\tilde{J}}\, A^{\tilde{I}}A^{\tilde{J}}\right) W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}e^{\sigma}\, \overset{\circ}{a}_{\tilde{I}\tilde{J}}\, F_{\mu\nu}^{\tilde{I}}F^{\mu\nu\,\tilde{J}} - e^{\sigma}\, \overset{\circ}{a}_{\tilde{I}\tilde{J}}\, A^{\tilde{I}}F_{\mu\nu}^{\tilde{J}}W^{\mu\nu} + \\ & + \frac{e^{-1}}{2\sqrt{6}}C_{\tilde{I}\tilde{J}\tilde{K}}\epsilon^{\mu\nu\rho\sigma}\left(A^{\tilde{I}}F_{\mu\nu}^{\tilde{J}}F_{\rho\sigma}^{\tilde{K}} + 2A^{\tilde{I}}A^{\tilde{J}}F_{\mu\nu}^{\tilde{K}}W_{\rho\sigma} + \frac{4}{3}A^{\tilde{I}}A^{\tilde{J}}A^{\tilde{K}}W_{\mu\nu}W_{\rho\sigma}\right), \end{aligned}$$

where $F_{\mu\nu}^{\tilde{I}} = 2\partial_{[\mu}A_{\nu]}^{\tilde{I}}$ are the Kaluza-Klein-invariant field strengths and $W_{\mu\nu} = 2\partial_{[\mu}W_{\nu]}$. The kinetic energy term of the scalar fields can be written as [9]

$$e^{-1}\mathcal{L}_S = -\frac{1}{2}\left(\tilde{a}_{\tilde{I}\tilde{J}}\partial_{\mu}\tilde{h}^{\tilde{I}}\partial^{\mu}\tilde{h}^{\tilde{J}} + \frac{2}{3}\tilde{a}_{\tilde{I}\tilde{J}}\partial_{\mu}A^{\tilde{I}}\partial^{\mu}A^{\tilde{J}}\right),$$

where we have defined $\tilde{a}_{\tilde{I}\tilde{J}} = \frac{3}{2}e^{-2\sigma}\, \overset{\circ}{a}_{\tilde{I}\tilde{J}}$, and $\tilde{h}^{\tilde{I}} = e^{\sigma}h^{\tilde{I}}$. We note that

$$C_{\tilde{I}\tilde{J}\tilde{K}}\tilde{h}^{\tilde{I}}\tilde{h}^{\tilde{J}}\tilde{h}^{\tilde{K}} = e^{3\sigma} > 0\,.$$

[14]In the rest of the paper we shall use hatted indices for the 5D vectors, i.e $\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3, 4$ and use unhatted indices for four-dimensional vectors, i.e $\mu, \nu, \ldots = 0, 1, 2, 3$.

4.1 Very special Kähler geometry and global symmetries

The above lagrangian can be recast in the language of (very) special Kähler geometry [35] by defining the complex coordinates [9],

$$z^{\tilde{I}} := \frac{1}{\sqrt{3}} A^{\tilde{I}} + \frac{i}{\sqrt{2}} \tilde{h}^{\tilde{I}} , \tag{4.3}$$

where

$$\tilde{h}^{\tilde{I}} := e^{\sigma} h^{\tilde{I}} . \tag{4.4}$$

These $(\tilde{n}+1)$ complex coordinates $z^{\tilde{I}}$ can be interpreted as the inhomogeneous coordinates corresponding to the $(\tilde{n}+2)$-dimensional complex vector[15]

$$X^{A} = \begin{pmatrix} X^{\check{0}} \\ X^{\tilde{I}} \end{pmatrix} = \begin{pmatrix} 1 \\ z^{\tilde{I}} \end{pmatrix} . \tag{4.5}$$

Introducing the "prepotential"

$$F(X^{A}) = -\frac{\sqrt{2}}{3} C_{\tilde{I}\tilde{J}\tilde{K}} \frac{X^{\tilde{I}} X^{\tilde{J}} X^{\tilde{K}}}{X^{\check{0}}} \tag{4.6}$$

and the symplectic section

$$\begin{pmatrix} X^{A} \\ F_{A} \end{pmatrix} \equiv \begin{pmatrix} X^{A} \\ \partial_{A} F \end{pmatrix} \tag{4.7}$$

one can define a Kähler potential

$$\begin{aligned} K(X, \bar{X}) &:= -\ln[i \bar{X}^{A} F_{A} - i X^{A} \bar{F}_{A}] && (4.8) \\ &= -\ln\left[i \frac{\sqrt{2}}{3} C_{\tilde{I}\tilde{J}\tilde{K}} (z^{\tilde{I}} - \bar{z}^{\tilde{I}})(z^{\tilde{J}} - \bar{z}^{\tilde{J}})(z^{\tilde{K}} - \bar{z}^{\tilde{K}}) \right] && (4.9) \end{aligned}$$

and a "period matrix"

$$\mathcal{N}_{AB} := \bar{F}_{AB} + 2i \frac{\mathrm{Im}(F_{AC}) \mathrm{Im}(F_{BD}) X^{C} X^{D}}{\mathrm{Im}(F_{CD}) X^{C} X^{D}} , \tag{4.10}$$

where $F_{AB} \equiv \partial_{A} \partial_{B} F$ etc. The particular ("very special") form (4.6) of the prepotential leads to

$$g_{\tilde{I}\bar{\tilde{J}}} \equiv \partial_{\tilde{I}} \partial_{\bar{\tilde{J}}} K = \frac{3}{2} e^{-2\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} \tag{4.11}$$

for the Kähler metric, $g_{\tilde{I}\bar{\tilde{J}}}$, on the scalar manifold $\mathcal{M}^{(4)}$ of the four-dimensional theory, and

$$\begin{aligned} \mathcal{N}_{\check{0}\check{0}} &= -\frac{2\sqrt{2}}{9\sqrt{3}} C_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{I}} A^{\tilde{J}} A^{\tilde{K}} - \frac{i}{3} \left(e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} A^{\tilde{I}} A^{\tilde{J}} + \frac{1}{2} e^{3\sigma} \right) && (4.12) \\ \mathcal{N}_{\check{0}\tilde{I}} &= \frac{\sqrt{2}}{3} C_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{J}} A^{\tilde{K}} + \frac{i}{\sqrt{3}} e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} A^{\tilde{J}} && (4.13) \\ \mathcal{N}_{\tilde{I}\tilde{J}} &= -\frac{2\sqrt{2}}{\sqrt{3}} C_{\tilde{I}\tilde{J}\tilde{K}} A^{\tilde{K}} - i e^{\sigma} \overset{\circ}{a}_{\tilde{I}\tilde{J}} && (4.14) \end{aligned}$$

[15] We label the top component of X^{A} as $\check{0}$ so as to distinguish it from the zeroth component X^{0} of $X^{\tilde{I}}$.

for the period matrix $\mathcal{N}_{AB}$. Defining

$$F^{\check{0}}_{\mu\nu} := -2\sqrt{3}W_{\mu\nu}\,, \tag{4.15}$$

the dimensionally reduced lagrangian (4.3) simplifies to

$$\begin{aligned} e^{-1}\mathcal{L}^{(4)} &= -\frac{1}{2}R - g_{\tilde{I}\bar{\tilde{J}}}(\partial_\mu z^{\tilde{I}})(\partial^\mu \bar{z}^{\tilde{J}}) + \\ &\quad + \frac{1}{4}\mathrm{Im}(\mathcal{N}_{AB})F^A_{\mu\nu}F^{\mu\nu B} - \frac{1}{8}\mathrm{Re}(\mathcal{N}_{AB})\epsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}F^B_{\rho\sigma}\,. \end{aligned} \tag{4.16}$$

The scalar fields $z^{\tilde{I}}$ are restricted to the domain $\mathcal{V}(Im(z)) > 0$. As was shown in [9], the scalar manifolds of the 4D, $\mathcal{N} = 2$ MESGTs obtained by dimensional reduction from five dimensions are simply the generalized "upper half planes" with respect to the cubic form for all the MESGTs. For those theories defined by euclidean Jordan algebras of degree 3 they are the Köcher upper half planes of the corresponding Jordan algebras. These are sometimes known as Siegel domains of the first kind. Cecotti showed that the more general class of homogeneous (not necessarily symmetric) scalar manifolds of 4D MESGTs are also Siegel domains of the first kind [36]. The upper half planes of the Jordan algebras can be mapped into bounded symmetric domain, which can be realized as hermitean symmetric spaces. For the Jordan algebras of degree 3, we list the scalar manifolds of the corresponding 4D MESGTs describing the coupling of $(\tilde{n}+1)$ vector multiplets to supergravity in the table below [9]:

$$\begin{aligned} J_3^{\mathbb{R}}: &\quad \mathcal{M} = \mathrm{Sp}(6,\mathbb{R})/\,\mathrm{U}(3) && (\tilde{n}+1=6) \\ J_3^{\mathbb{C}}: &\quad \mathcal{M} = \mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3)) && (\tilde{n}+1=9) \\ J_3^{\mathbb{H}}: &\quad \mathcal{M} = \mathrm{SO}^*(12)/\,\mathrm{U}(6) && (\tilde{n}+1=15) \\ J_3^{\mathbb{O}}: &\quad \mathcal{M} = E_{7(-25)}/E_6\times\mathrm{U}(1) && (\tilde{n}+1=27) \\ J=(\mathbb{R}\oplus\Sigma_{\tilde{n}}): &\quad \mathcal{M} = \frac{\mathrm{SO}(\tilde{n},2)\times\mathrm{SO}(2,1)}{\mathrm{SO}(\tilde{n})\times\mathrm{SO}(2)\times\mathrm{SO}(2)} && (\tilde{n}+1)\,. \end{aligned} \tag{4.17}$$

In [37], isometries of general homogeneous manifolds of very special real, Kähler and quaternionic type were studied. More specifically , the authors of [37] first classified very special real scalar manifolds of 5D, $\mathcal{N}=2$ MESGTs that are homogeneous , but not necessarily symmetric. They then studied the isometries of very special Kähler and quaternionic spaces that follow from the dimensional reduction of those $5D$ theories to four and three dimensions, respectively. They confirmed and generalized the earlier results showing that there is a map, **r**, from very special real manifolds of dimension $\tilde{n}$ to very special Kähler manifolds of complex dimension $\tilde{n}+1$, and a map, **c**, from special Kähler manifolds of complex dimension $\tilde{n}+1$ to special quaternionic manifolds of quaternionic dimension $\tilde{n}+2$. This phenomenon was first observed for theories originating from 5D MESGTs defined by Jordan algebras of degree 3 with symmetric target spaces [9].

Upon dimensional reduction from five dimensions, the resulting lagrangian is still invariant under the action of the symmetry group, G, of $C_{\tilde{I}\tilde{J}\tilde{K}}$. Its action on the complex scalars $z^{\tilde{I}}$ parameterizing the Kähler manifold is simply

$$\delta z^{\tilde{I}} = B^{\tilde{I}}{}_{\tilde{J}}z^{\tilde{J}}\,,$$

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which leaves the Kähler potential $K(z,\bar z) = -\ln \mathcal{V}(z-\bar z) + \text{const.}$, invariant. The vector fields A_μ^A transform in a reducible representation under G, with the components transforming as

$$\begin{aligned} \delta A_\mu^{\tilde I} &= B^{\tilde I}{}_{\tilde J} A_\mu^{\tilde J} \\ \delta W_\mu &= 0\,. \end{aligned}$$

Under shifts of the scalars $A^{\tilde I} \to A^{\tilde I} + b^{\tilde I}$, which come from the vector fields in five dimensions, the Kähler potential is manifestly invariant. Furthermore, one finds that under *real constant* shifts of the scalar fields

$$z^{\tilde I} \to z^{\tilde I} + b^{\tilde I},$$

together with the shifts

$$F_{\mu\nu}^{\tilde I} \to F_{\mu\nu}^{\tilde I} - 2W_{\mu\nu} b^{\tilde I}$$

the reduced lagrangian is also invariant up to surface terms which are interpreted as shifts of the theta angle of the $4D$ supersymmetric theory. Furthermore, for any dimensionally reduced supergravity theory, there is also a scaling symmetry $\mathrm{SO}(1,1)$, whose generator is denoted as β. It acts as $X^{\check 0} \to e^{\beta} X^{\check 0}, \quad X^{\tilde I} \to e^{-\frac{\beta}{3}} X^{\tilde I}$.

The authors of [37] also elucidated the conditions under which the additional, "hidden", isometries, with infinitesimal parameters $a^{\tilde I}$, exist which do not have a clear origin from the dimensional reduction of the $5D$ theories. The number of hidden isometries is determined by the maximal number of non-zero parameters in $a_0, \ldots, a_n$ that satisfy

$$a_{\tilde M} E^{\tilde M}_{\tilde I \tilde J \tilde K \tilde L} = 0\,, \tag{4.18}$$

where

$$E^{\tilde M}_{\tilde I \tilde J \tilde K \tilde L} := \frac{-27}{(\mathcal{V}(\tilde h))^2} C^{\tilde E \tilde F \tilde M} C_{\tilde E (\tilde I \tilde J} C_{\tilde K \tilde L)\tilde F} - \delta^{\tilde M}_{(\tilde I} C_{\tilde J \tilde K \tilde L)}$$

and

$$C^{\tilde E \tilde F \tilde M} \equiv \overset{\circ}{a}{}^{\tilde E \tilde I} \overset{\circ}{a}{}^{\tilde F \tilde J} \overset{\circ}{a}{}^{\tilde M \tilde K} C_{\tilde I \tilde J \tilde K} \tag{4.19}$$

$$\mathcal{V}(\tilde h) \equiv C_{\tilde I \tilde J \tilde K} \tilde h^{\tilde I} \tilde h^{\tilde J} \tilde h^{\tilde K} = e^{3\sigma}\,. \tag{4.20}$$

We should note that whereas $C_{\tilde I \tilde J \tilde K}$ are constant, $C^{\tilde I \tilde J \tilde K}$ are scalar field dependent in general. As was shown in [37] all the isometries of the scalar manifold extend to the full generalized duality group of the MESGT in four dimensions.

The Lie algebra of the full set of isometries has a three-graded decomposition with respect to the eigenvalues of the scale symmetry generator β :

$$\mathcal{W} = \mathcal{W}^{-1} \oplus \mathcal{W}^{0} \oplus \mathcal{W}^{+1}\,, \tag{4.21}$$

where $\mathcal{W}^0$ consists of the generators of $B^{\tilde I}{}_{\tilde J}$ and β, $\mathcal{W}^{-1}$ is associated with the real parameters $a^{\tilde I}$, and $\mathcal{W}^{+1}$ is associated with the real parameters $b^{\tilde I}$.

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It was originally stated in [21] that the subalgebra

$$\mathcal{W}^0 \oplus \mathcal{W}^{+1}$$

is the maximal symmetry subalgebra of the full duality group under which the lagrangian is invariant and, hence, to obtain $\mathcal{N}=2$ YMESGTs one may then try to gauge subgroups of this parabolic subgroup. However this statement is true only for the symplectic section of the theory that comes from direct dimensional reduction [37]. For example, for symmetric scalar manifolds the full duality group has negative grade generators and, via dualizations of the vector fields, one can go to different symplectic sections of the respective $\mathcal{N}=2$ MESGT such that the lagrangian becomes invariant under other subgroups of the full duality group generated by $\mathcal{W}$ that are not subgroups of the parabolic group generated by $\mathcal{W}^0 \oplus \mathcal{W}^{+1}$ as will be discussed in the next section.

The Lie algebras $\mathcal{W}$ of the isometries of the symmetric scalar manifolds of four-dimensional theories obtained from those 5D theories defined by euclidean Jordan algebras of degree three can be identified with the Lie algebras of the conformal (or "linear fractional") groups of the corresponding Jordan algebras. As explained in appendix A, the conformal groups of Jordan algebras are generated by translations, $\mathfrak{T}_J$, special conformal generators, $\mathfrak{K}_J$, and the generators of the structure group (which is simply the direct product of the generalized "Lorentz group" of the Jordan algebra and dilatations, as explained in appendix A). The conformal Lie algebra of J has a 3-graded decomposition with respect its structure algebra

$$\mathrm{Conf}(J) = \mathfrak{T}_J \oplus \mathfrak{str}(J) \oplus \mathfrak{K}_J \,,$$

which coincides with the three grading of $\mathcal{W}$ (eq. (4.21)) by identification of the dilatation generator with β. The Köcher upper half-space $(J+iJ_+)$ of a Jordan algebra is spanned by complexified elements of the form $X+iY$ where $X, Y \in J$ such that Y has positive norm. The upper half-space of a euclidean Jordan algebra J can be mapped into the hermitean symmetric space [38]

$$(J+iJ_+) \iff \frac{\mathrm{Conf}(J)}{K(J)} \,,$$

where $K(J)$ is the subgroup generated by the derivations ("rotations") and generators of the form $(\mathfrak{T}_J - \mathfrak{K}_J)$. For euclidean Jordan algebras $K(J)$ turns out to be the maximal compact subgroup of $\mathrm{Conf}(J)$. For the MESGTs defined by euclidean Jordan algebras of degree three, the corresponding symmetric spaces were listed in (4.17).

The scalar manifolds, $\mathcal{M}_4$, of the four-dimensional theories obtained by dimensionally reducing a 5D unified MESGT defined by lorentzian Jordan algebras are, in general, the "upper half-spaces of cubic hypersurfaces" defined by $d_{\tilde{I}\tilde{J}\tilde{K}}$. The complex coordinates of $\mathcal{M}_4$ can be written as

$$z^{\tilde{I}} = \sqrt{\frac{1}{2}}\left(\sqrt{\frac{2}{3}}A^{\tilde{I}} + i\tilde{h}^{\tilde{I}}\right)$$

such that

$$d_{\tilde{I}\tilde{J}\tilde{K}}\tilde{h}^{\tilde{I}}\tilde{h}^{\tilde{J}}\tilde{h}^{\tilde{K}} > 0 \,,$$

J	$\mathrm{Conf}(J)$	$\mathrm{K}(J)$
$J^{\mathbb{R}}_{(1,N)}$	$\mathrm{Sp}(2N+2,\mathbb{R})$	$\mathrm{SU}(N,1)\times \mathrm{U}(1)$
$J^{\mathbb{C}}_{(1,N)}$	$\mathrm{SU}(N+1,N+1)$	$\mathrm{SU}(N,1)\times \mathrm{SU}(N,1)\times \mathrm{U}(1)$
$J^{\mathbb{H}}_{(1,N)}$	$\mathrm{SO}^*(4N+4)$	$\mathrm{SU}(2N,2)\times \mathrm{U}(1)$
$J^{\mathbb{O}}_{(1,2)}$	$E_{7(-25)}$	$E_{6(-14)}\times \mathrm{U}(1)$

Table 2: List of the simple lorentzian Jordan algebras of type $J^{\mathbb{A}}_{(1,N)}$ and their conformal (Möbius) groups and their subgroups $K(J)$.

J	$\mathrm{Aut}(J)ⓈT_J$	$\mathcal{M}_4$
$J^{\mathbb{R}}_{(1,3)}$	$\mathrm{SO}(3,1)ⓈT_9$	$\mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$
$J^{\mathbb{C}}_{(1,3)}$	$\mathrm{SU}(3,1)ⓈT_{15}$	$\mathrm{SO}^*(12)/\mathrm{U}(6)$
$J^{\mathbb{H}}_{(1,3)}$	$\mathrm{USp}(6,2)ⓈT_{27}$	$E_{7(-25)}/E_6\times \mathrm{U}(1)$

Table 3: Simple lorentzian Jordan algebras of type $J^{\mathbb{A}}_{(1,3)}$ for $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$. The second column lists the manifest isometries and third column lists the scalar manifolds of $\mathcal{N}=2$ MESGTs in 4D defined by them.

with its metric given by equation (4.11). Formally we can write

$$\mathcal{M}_4 \iff A+i\tilde{h}\,,$$

where $A=e_{\tilde{I}}A^{\tilde{I}}$ and $\tilde{h}=e_{\tilde{I}}\tilde{h}^{\tilde{I}}$ are traceless elements of the lorentzian Jordan algebra such that A is unconstrained otherwise and $\tilde{h}$ has positive cubic norm. From this it is evident that the scalar manifold $\mathcal{M}_4$ can be mapped into a subspace of the Koecher upper half space of the full Jordan algebra $J^{\mathbb{A}}_{(1,N)}$ spanned by elements of the form $X+iY$ where X and Y are general elements (not necessarily traceless) of the Jordan algebra $J^{\mathbb{A}}_{(1,N)}$ such that the standard norm $\mathcal{P}(Y)$ of degree $N+1$ of the element Y is positive. The upper half-planes of the Jordan algebras $J^{\mathbb{A}}_{(1,N)}$, with an appropriately defined metric, can be identified with the coset spaces of the form

$$\frac{\mathrm{Conf}(J^{\mathbb{A}}_{(1,N)})}{\mathrm{K}(J^{\mathbb{A}}_{(1,N)})}\,.$$

In table 2 we give the list of the groups that underlie these spaces for all lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$.

As was discussed in section 2, the cubic form defined over the traceless elements of the lorentzian Jordan algebras $J^{\mathbb{R}}_{(1,3)},J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ coincide with the generic cubic norms of degree three euclidean Jordan algebras $J^{\mathbb{C}}_3,J^{\mathbb{H}}_3$ and $J^{\mathbb{O}}_3$, respectively. Hence for the $\mathcal{N}=2$ MESGTs defined by $J^{\mathbb{R}}_{(1,3)},J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ the scalar manifolds are the symmetric spaces listed in table 3.

5. Unified MESGTs and YMESGTs in four dimensions

In this section, we will give a classification of four-dimensional unified $\mathcal{N}=2$ MESGTs and

YMESGTs. Since a full classification of four-dimensional MESGTs with non-homogeneous scalar manifolds and non-trivial isometry groups does not yet exist we will restrict our classification to theories with homogeneous scalar manifolds and those theories with non-homogeneous manifolds defined by lorentzian Jordan algebras. As mentioned earlier, the analysis in four dimensions is more complicated than in five due to the fact that the full U-duality group, $\mathcal{U}$, is an on-shell symmetry and not the symmetry of the lagrangian in four dimensions. The electric field strengths $\mathcal{F}^{A\,\mu\nu}$ together with the magnetic (dual) field strengths $\mathcal{G}_{A\,\mu\nu}$ form a representation of the U-duality group $\mathcal{U}$, which may be reducible. We shall call a MESGT in 4D unified if the electric field strengths $\mathcal{F}^{A\,\mu\nu}$, including that of the graviphoton, form an irreducible representation of a simple symmetry group G of the lagrangian. Since, in 4D, massless vector fields are dual to other vector fields, we have some freedom in choosing electric field strengths versus the magnetic field strengths. By dualizing some of the vector fields one can obtain some *real* linear combinations of the ($\mathcal{F}^{A\,\mu\nu}$ and $*(\mathcal{G})_{A\,\mu\nu}$) to be the new electric field strengths. In so doing one also changes the subgroup G of the full U-duality group that is a symmetry of the lagrangian. This phenomenon can best be explained by the example of maximal supergravity. The five-dimensional $\mathcal{N}=8$ supergravity with U-duality group $E_{6(6)}$ yields, upon dimensional reduction to four dimensions, the parabolic group

$$[E_{6(6)} \times \mathrm{SO}(1,1)] Ⓢ T_{27}$$

as the symmetry of the lagrangian.[16] However we know that by dualization one can obtain $\mathrm{SL}(8,\mathbb{R})$ [39] or $\mathrm{SU}^*(8)$ [40, 41] as the symmetry of the lagrangian. We should stress that both $\mathrm{SL}(8,\mathbb{R})$ and $\mathrm{SU}^*(8)$ are subgroups of the full U-duality group $E_{7(7)}$ under which electric and magnetic fields transform in $\mathbf{28}$ and $\tilde{\mathbf{28}}$, which are real representations of $\mathrm{SL}(8,\mathbb{R})$ and $\mathrm{SU}^*(8)$, respectively. Even though SU(8) is also a subgroup of $E_{7(7)}$ under which $\mathbf{56} = \mathbf{28} \oplus \overline{\mathbf{28}}$, it can not be made a symmetry of the lagrangian by dualization, since 28 is a complex representation of SU(8) and would thus require complex vector potentials.

One should also note that, in contrast to the situation in five dimensions, neither the irreducibility of the representation $(\mathcal{F}^{A\,\mu\nu} \oplus \mathcal{G}_{A\,\mu\nu})$ under $\mathcal{U}$ nor the simplicity of the group $\mathcal{U}$ is necessary or sufficient for having a unified MESGT in four dimensions. This will also be illustrated in some of our examples later in the text.

We will begin our analysis by studying the symmetries of *pure* 5D, $\mathcal{N}=2$ supergravity theories dimensionally reduced to four dimensions and its gaugings.

5.1 Pure 5D, $\mathcal{N}=2$ supergravity dimensionally reduced to 4D

Since the symmetry group structure of dimensionally reduced pure supergravity will be helpful in understanding the action of U-duality group in the general case we shall review it first. The dimensional reduction of pure 5D, $\mathcal{N}=2$ supergravity to 4D was first studied long time ago in [42] and more recently in [43]. The resulting theory is $\mathcal{N}=2$, 4D supergravity coupled to one vector multiplet and the scalar manifold is $\mathrm{SU}(1,1)/\mathrm{U}(1)$, where the two scalars come from the graviton and the graviphoton in five dimensions.

[16]The symbol Ⓢ denotes semidirect product and T_{27} denotes the translation group in 27 dimensions.

The manifest symmetry of the dimensionally reduced lagrangian is $\mathrm{SO}(1,1)_\beta \circledS T$ where T represents the translations of the scalar coming from the graviphoton in five dimensions and the generator of scale transformations $\mathrm{SO}(1,1)_\beta$ is β. The "hidden" isometry of the reduced theory corresponds to special conformal transformations K that extends $\mathrm{SO}(1,1)_\beta \circledS T$ to $\mathrm{SU}(1,1)$. The compact $U(1)$ generator of $\mathrm{SU}(1,1)$ is $\mathfrak{t} - \mathfrak{k}$, where $\mathfrak{t}$ and $\mathfrak{k}$ are generators of translations and special conformal transformations, respectively. The $\mathrm{SU}(1,1)$ acts as the full U-duality group on-shell. Remarkably, the two vector field strengths and their duals transform in the irreducible four-dimensional representation of $\mathrm{SU}(1,1)$ [43] (In the language of compact $\mathrm{SU}(2)$ it is the spin $3/2$ representation of $\mathrm{SU}(1,1)$). What this means is that all the field strengths and their duals carry a charge under any abelian subgroup of $\mathrm{SU}(1,1)$, be it $U(1)$ or $\mathrm{SO}(1,1)$. Since the gauge field of an abelian gauge theory must be neutral under the abelian group, one can not gauge any abelian subgroup of $\mathrm{SU}(1,1)$. We shall denote the full U-duality group of dimensionally reduced pure $\mathcal{N}=2$ supergravity as $\mathrm{SU}(1,1)_G$. All other $\mathcal{N}=2$ MESGTs dimensionally reduced to 4D can be truncated to this theory. $\mathrm{SU}(1,1)_G$ will be a subgroup of the full U-duality groups of dimensionally reduced MESGTs defined by euclidean Jordan algebras of degree three , whose scalar manifolds are symmetric spaces. However, in general, only the $\mathrm{SO}(1,1) \circledS T$ subgroup of $\mathrm{SU}(1,1)_G$ will be a symmetry of dimensionally reduced MESGTs as will become evident when we discuss theories whose scalar manifolds are homogeneous but not symmetric.

5.2 Theories whose scalar manifolds are symmetric spaces

- The generic Jordan family with target manifolds

$$\frac{\mathrm{SO}(\tilde{n}-1,1)}{\mathrm{SO}(\tilde{n}-1)} \times \mathrm{SO}(1,1)$$

in 5D lead to four dimensional MESGTs with target manifolds [9]

$$\mathcal{M} = \frac{\mathrm{SO}(\tilde{n},2)}{\mathrm{SO}(\tilde{n}) \times \mathrm{SO}(2)} \times \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\,. \tag{5.1}$$

Under the U-duality group, $\mathcal{U} = \mathrm{SO}(\tilde{n},2) \times \mathrm{SU}(1,1)$, the $(\tilde{n}+2)$ field strengths $\mathcal{F}^{A\,\mu\nu}$ and their magnetic duals $\mathcal{G}_{A\,\mu\nu}$ transform in the $(\tilde{n}+2, 2)$ representation. One can take a symplectic section of these theories such that all the electric field strengths $\mathcal{F}^{A\,\mu\nu}$ transform in the real irreducible $(\tilde{n}+2)$ representation of the simple subgroup $\mathrm{SO}(\tilde{n},2)$ of $\mathcal{U}$. Thus the generic Jordan family of $\mathcal{N}=2$ MESGTs in 4D are unified theories, in contrast to their five-dimensional counterparts. To obtain a unified YMESGT by gauging a simple subgroup H of $\mathrm{SO}(\tilde{n},2)$, all the $(\tilde{n}+2)$ electric field strengths must transform in the adjoint representation of H. This is only possible for $\tilde{n}=1$. Thus the unique unified YMESGT in the generic Jordan family has $\mathrm{SO}(2,1)$ as its simple gauge group.[17]

[17] Interestingly, this unified YMESGT forms a common subsector of several of the models with stable de Sitter vacua considered in [44]. However, it is really only a subsector, and additional vector multiplets and gauge group factors are necessary in order to obtain stable de Sitter vacua [44]. Clearly, with these additional gauge group factors, these theories as a whole are then no longer unified YMESGTs.

We should stress that the $\mathrm{SU}(1,1) = \mathrm{SO}(2,1)$ factor in the four dimensional U-duality group is not the $\mathrm{SU}(1,1)_G$ symmetry of the pure supergravity dimensionally reduced to four dimensions. This is evident from the fact that the field strengths and their duals form 2 dimensional (spin 1/2) representations of the former. The $\mathrm{SU}(1,1)_G$ is a diagonal subgroup of $\mathrm{SU}(1,1)$ factor and an $\mathrm{SO}(2,1)$ subgroup of $\mathrm{SO}(\tilde{n},2)$ under which we have the following decompositions:

$$\begin{aligned}
&\mathrm{SO}(2,1) \times \mathrm{SO}(\tilde{n},2) \supset \mathrm{SO}(2,1) \times \mathrm{SO}(2,1) \times \mathrm{SO}(\tilde{n}-1) \supset \mathrm{SU}(1,1)_G \times \mathrm{SO}(\tilde{n}-1)\\
&(2,\tilde{n}+2) = (2,3,1) \oplus (2,1,\tilde{n}-1) = (4,1) \oplus (2,1) \oplus (2,\tilde{n}-1)\,.
\end{aligned}$$

Note that the centralizer of $\mathrm{SU}(1,1)_G$ inside the four-dimensional U-duality group is the automorphism group of the corresponding Jordan algebra, which is $\mathrm{SO}(\tilde{n}-1)$. This is a general feature of all theories defined by Jordan algebras of degree three as will be made evident below.

Let us next analyze the magical supergravity theories defined by *simple* Jordan algebras of degree 3, whose scalar manifolds are all symmetric.

- For the $\mathcal{N}=2$ MESGT defined by $J_3^{\mathbb{R}}$, the four-dimensional U-duality group is $\mathrm{Sp}(6,\mathbb{R})$, which is also the isometry group of its scalar manifold,

$$\mathcal{M} = \frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)}\,. \tag{5.2}$$

  Under the U-duality group $\mathrm{Sp}(6,\mathbb{R})$, the $(\tilde{n}+2) = 7$ electric field strengths of this theory together with their magnetic duals transform in the irreducible 14-dimensional representation. However, $\mathrm{Sp}(6,\mathbb{R})$ does not have a simple subgroup G under which all the seven electric field strengths transform irreducibly. Thus the four-dimensional $\mathcal{N}=2$ theory defined by $J_3^{\mathbb{R}}$ is *not* a unified MESGT. This provides an example of a theory with a simple U-duality group, $\mathcal{U}$, under which electric and magnetic field strengths transform in an irrep of $\mathcal{U}$, but which is nevertheless not unified by our definition. Consequently, it is also *not* possible to gauge this theory so as to obtain a unified YMESGT.

  The centralizer of $\mathrm{SU}(1,1)_G$ inside the four-dimensional U-duality group $\mathrm{Sp}(6,\mathbb{R})$ is $\mathrm{SO}(3)$ which is the automorphism group of $J_3^{\mathbb{R}}$:

$$\begin{aligned}
&\mathrm{Sp}(6,\mathbb{R}) \supset \mathrm{SU}(1,1)_G \times \mathrm{SO}(3)\\
&14 = (4,1) + (2,5)\,.
\end{aligned}$$

- The MESGT defined by the Jordan algebra $J_3^{\mathbb{C}}$ has the U-duality group $\mathrm{SU}(3,3)$ in 4D, under which the field strengths and their magnetic duals, $\mathcal{F}^{A\,\mu\nu} \oplus \mathcal{G}_{A\,\mu\nu}$, transform in the irreducible symplectic representation **20**. The group $\mathrm{SU}(3,3)$ has a subgroup under which the 20 decomposes into two real ten-dimensional irreps, namely, $SO(3,3) \cong SL(4,\mathbb{R})$.[18]

$$\begin{aligned} SU(3,3) &\supset SL(4,\mathbb{R}) \\ 20 &= 10 + \tilde{10}\,. \end{aligned}$$

Therefore, one can take a symplectic section of this theory such that $SL(4,\mathbb{R})$ becomes the electric subgroup of the full U-duality group. Hence this theory defined by $J_3^{\mathbb{C}}$ is a unified MESGT. Furthermore, one can gauge this theory to obtain a unified YMESGT with the gauge group $SO(3,2) \cong Sp(4,\mathbb{R})$, since under the $SO(3,2)$ subgroup all the electric field strengths transform in the adjoint representation.

The centralizer of $SU(1,1)_G$ inside $SU(3,3)$ is $SU(3)$

$$\begin{aligned} SU(3,3) &\supset SU(1,1)_G \times SU(3) \\ 20 &= (4,1) + (2,8)\,. \end{aligned}$$

- The theory defined by the quaternionic Jordan algebra $J_3^{\mathbb{H}}$ has the U-duality group $SO^*(12)$ in four dimensions under which the field strengths of the vector fields together with their duals transform in the spinor representation **32**. The maximal rank subgroups of $SO^*(12)$ are

$$\begin{aligned} SO^*(12) &\supset SO(6,\mathbb{C}) \cong SL(4,\mathbb{C}) \\ SO^*(12) &\supset SO^*(2p) \times SO^*(12-2p)\,, \qquad p = 1,2,3 \\ SO^*(12) &\supset U(6) \\ SO^*(12) &\supset SU(p,6-p) \times U(1) \\ SO^*(12) &\supset SU^*(6) \times SO(1,1) \end{aligned} \tag{5.3}$$

Under the $SO^*(10) \times SO(2)$ subgroup the spinor **32** of $SO^*(12)$ decomposes as $\mathbf{16}^{+1} \oplus \overline{\mathbf{16}}^{-1}$. The spinor representation 16 of $SO^*(10)$ is complex.[19] However, under the subgroup $SO(6,\mathbb{C}) \cong SL(4,\mathbb{C})$, the spinor representation 32 of $SO^*(12)$ decomposes as

$$32 = (4,\bar{4}) \oplus (\bar{4},4)\,.$$

Since the $(4,\bar{4})$ representation of $SL(4,\mathbb{C})$ is real, the corresponding MESGT is a unified theory. The only simple noncompact group of dimension 16 is $SL(3,\mathbb{C})$, which is a subgroup of $SL(4,\mathbb{C})$. However, $(4,\bar{4})$ of $SL(4,\mathbb{C})$ does not transform irreducibly under the $SL(3,\mathbb{C})$ subgroup and hence this theory can not be gauged so as to obtain a unified YMESGT in 4D.

We should recall that the corresponding theory in five dimensions is a unified MESGT and can be gauged [6] so as to obtain the unique unified YMESGT with gauge group $SO^*(6) = SU(3,1)$ in 5D whose scalar manifold is a symmetric space.

[18] Under the $SU(3,2)$ subgroup 20 also decomposes as $10 \oplus \bar{10}$. However they are complex.

[19] $SO^*(10)$ does not admit any Majorana-Weyl spinors.

For this four-dimensional theory the centralizer of $\mathrm{SU}(1,1)_G$ is $\mathrm{USp}(6)$

$$\begin{aligned}&\mathrm{SO}^*(12) \supset \mathrm{SU}(1,1) \times \mathrm{Usp}(6)\\&32 = (4,1) + (2,14)\,.\end{aligned}$$

- The exceptional supergravity theory defined by the Jordan algebra $J_3^{\mathbb{O}}$ in four dimensions has the U-duality group $E_{7(-25)}$ with the maximal compact subgroup $E_6 \times U(1)$. The 28 electric field strengths together with their magnetic duals transform in the 56 dimensional irrep of $E_{7(-25)}$. The interesting subgroups of $E_{7(-25)}$ are

$$\begin{aligned}&E_{7(-25)} \supset \mathrm{SU}^*(8)\\&E_{7(-25)} \supset \mathrm{SU}(6,2)\\&E_{7(-25)} \supset \mathrm{SO}(10,2) \times \mathrm{SO}(2,1)\\&E_{7(-25)} \supset \mathrm{SO}^*(12) \times \mathrm{SU}(2)\\&E_{7(-25)} \supset E_{6(-26)} \times \mathrm{SO}(1,1).\end{aligned} \tag{5.4}$$

Under $\mathrm{SO}(10,2) \times \mathrm{SO}(2,1)$ and $\mathrm{SO}^*(12) \times \mathrm{SU}(2)$ the 56 of $E_{7(-25)}$ decomposes as

$$56 = (12,2) + (32,1)\,,$$

and hence $\mathrm{SO}(10,2)$ or $\mathrm{SO}^*(12)$ cannot act irreducibly on all the electric field strengths. The same holds true for $E_{6(-26)}$ under which the 56 of $E_{7(-25)}$ decomposes as

$$56 = 1 + 27 + \overline{27} + 1\,.$$

On the other hand, under the $\mathrm{SU}^*(8)$ or $\mathrm{SU}(6,2)$ subgroup the 56 of $E_{7(-25)}$ decomposes as

$$56 = 28 + \tilde{28}\,.$$

Now, the 28-dimensional representation of $\mathrm{SU}(6,2)$ is complex and hence $\mathrm{SU}(6,2)$ can not be a symmetry of the lagrangian. On the other hand, the 28 is a real representation of $\mathrm{SU}^*(8)$ and one can choose a symplectic section of the theory via dualization such that $\mathrm{SU}^*(8)$ becomes a symmetry of the lagrangian with all the electric field strengths transforming irreducibly in the 28. Thus, the exceptional MESGT is a unified MESGT. Furthermore, one can gauge the $\mathrm{SO}^*(8) = \mathrm{SO}(6,2) = \mathrm{SU}^*(8)$ subgroup so as to obtain a unified YMESGT. $\mathrm{SO}^*(8) = \mathrm{SO}(6,2)$ is the common maximal subgroup of $\mathrm{SU}(6,2)$ and $\mathrm{SU}^*(8)$.

We should note the important fact that the $\mathrm{SO}(6,2)$ gauged exceptional supergravity is the $\mathcal{N} = 2$ supersymmetric analog of the $\mathrm{SO}(6,2)$ gauged maximal $\mathcal{N} = 8$ supergravity [45], where $\mathrm{SO}(6,2)$ is the common subgroup of the $\mathrm{SU}(6,2)$ and $\mathrm{SU}^*(8)$ subgroups of $E_{7(7)}$. This extends further the parallels between the exceptional $\mathcal{N} = 2$ supergravity and the maximal supergravity. They both have different real forms of E_6, E_7 and E_8 as their U-duality groups in 5, 4 and 3 dimensions, respectively. Both theories can be gauged in five dimensions with the noncompact gauge group $\mathrm{SU}(3,1)$

$\mathcal{M}$	Unified global symmetry	Unified local symmetry
$\frac{\mathrm{SO}(\tilde{n},2)}{\mathrm{SO}(\tilde{n})\times\mathrm{SO}(2)}\times\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\mathrm{SO}(\tilde{n},2)$	$\mathrm{SO}(1,2)\quad(\tilde{n}=1)$
$\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)}$	–	–
$\frac{\mathrm{SU}(3,3)}{\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3))}$	$\mathrm{SL}(4,\mathbb{R})$	$\mathrm{SO}(3,2)$
$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}$	$\mathrm{SL}(4,\mathbb{C})$	–
$\frac{E_{7(-25)}}{E_6\times\mathrm{U}(1)}$	$\mathrm{SU}^*(8)$	$\mathrm{SO}(6,2)$

Table 4: List of the very special 4D MESGTs with symmetric target spaces. The second column displays the possible unified global symmetries, and the third column lists the possible unified local symmetry groups.

and twelve tensor fields [46, 23]. They both can be gauged in four dimensions with the gauge group $\mathrm{SO}(6,2)$.

For the exceptional theory the centralizer of $\mathrm{SU}(1,1)_G$ is F_4

$$\begin{aligned} &E_{7(-25)} \supset \mathrm{SU}(1,1)_G \times F_4 \\ &56 = (4,1) + (2,26)\,. \end{aligned}$$

We summarize our results for the symmetric space theories in table 4.

5.3 Theories with homogeneous, but non-symmetric scalar manifolds

The generic non-Jordan family of MESGTs in 5D have scalar manifolds of the form

$$\frac{\mathrm{SO}(\tilde{n},1)}{\mathrm{SO}(\tilde{n})}$$

describing the coupling of $\tilde{n}$ vector multiplets to $\mathcal{N}=2$ supergravity [25]. As was shown in [34], the corresponding $C_{\tilde{I}\tilde{J}\tilde{K}}$ tensor, and hence the lagrangian is invariant only under the subgroup $[\mathrm{SO}(\tilde{n}-1)\times\mathrm{SO}(1,1)_\lambda]ⓈT_{(\tilde{n}-1)}$ of $\mathrm{SO}(\tilde{n},1)$. Since, under the action of the maximal simple subgroup $\mathrm{SO}(\tilde{n}-1)$, two of the vectors transform as singlets, the generic non-Jordan family of MESGTs are not unified in 5D.

As a consequence of the fact that the full isometry group $\mathrm{SO}(\tilde{n}-1)$ is broken to its subgroup $[\mathrm{SO}(\tilde{n}-1)\times\mathrm{SO}(1,1)]ⓈT_{(\tilde{n}-1)}$ by the interaction terms of the 5D lagrangian, the scalar manifolds of the corresponding dimensionally reduced MESGTs in 4D are homogeneous, but not symmetric spaces. The Lie algebras of their isometry groups are not simple and have the following non semisimple Lie algebra graded with respect to the $\mathrm{SO}(1,1)_\alpha$ generator $\alpha = \frac{2}{3}\lambda + \beta$ [37]

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2\,,$$

where

$$\begin{aligned} \mathfrak{g}^0 &= \alpha \oplus \mathfrak{so}(2,1) \oplus \mathfrak{so}(\tilde{n}-1) \\ \mathfrak{g}^1 &= (1,2,\tilde{n}-1) \\ \mathfrak{g}^2 &= (2,0,0)\,. \end{aligned}$$

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The semisimple subgroup of the isometry group is $\mathrm{SO}(2,1)\times\mathrm{SO}(\tilde{n}-1)$ under which the $\tilde{n}+2$ electric field strengths decompose as $(3,1)\oplus(1,\tilde{n}-1)$. Thus the corresponding MESGT's are not unified.

Let us now consider the $4D$ MESGTs that are obtained from $5d$ theories whose scalar manifolds (in five dimensions) are homogeneous, but not symmetric. A complete list of possible scalar manifolds of $5d$ MESGTs that are homogeneous spaces was given in [24]. To achieve this classification the authors of [24] showed that the requirement of a transitive isometry group allows one to transform the most general solution for the symmetric tensor from the "canonical basis" to the basis:

$$\begin{aligned} C_{011} &= 1 \\ C_{0\bar{\mu}\bar{\nu}} &= -\delta_{\bar{\mu}\bar{\nu}} \\ C_{1\bar{i}\bar{j}} &= -\delta_{\bar{i}\bar{j}} \\ C_{\bar{\mu}\bar{i}\bar{j}} &= \gamma_{\bar{\mu}\bar{i}\bar{j}}\,, \end{aligned} \tag{5.5}$$

where the indices $\tilde{I}$ are now split such that $\tilde{I}=0,1,\bar{\mu},\bar{i}$ with $\bar{\mu}=1,2,\ldots,q+1$ and $\bar{i}=1,2,\ldots,r$ $(\tilde{n}=r+q+2)$. The coefficients $\gamma_{\bar{\mu}\bar{i}\bar{j}}$ are $(q+1)$ real $r\times r$ matrices that generate a real Clifford algebra, $\mathcal{C}(q+1,0)$, of positive signature. The allowed homogeneous (but not symmetric) spaces are, in general, quotients of "parabolic groups" G modded out by their maximal compact subgroups H. The Lie algebra $\mathfrak{g}$ of the group G is a semi-direct sum:

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} \tag{5.6}$$

$$\begin{aligned} \mathfrak{g}^0 &= \mathfrak{so}_\lambda(1,1)\oplus\mathfrak{so}(q+1,1)\oplus\mathcal{S}_q(P,Q) \\ \mathfrak{g}^{+1} &= (s,v)\,. \end{aligned} \tag{5.7}$$

Here, $\mathcal{S}_q(P,Q)$ is a Lie algebra generated by the metric preserving elements of the centralizer of the Clifford algebra representation, whereas s denotes a spinor representation of $so(q+1,1)$ (of dimension $\mathcal{D}_{q+1}$), and v denotes the vector representation of $\mathcal{S}_q(P,Q)$, which is of dimension $(P+Q)$.[20] The isotropy group H is

$$H=\mathrm{SO}(q+1)\otimes\mathcal{S}_q(P,Q)\,. \tag{5.8}$$

The possible groups $\mathcal{S}_q(P,Q)$ and the associated real Clifford algebras were given in [24] which we list in table 5.

Under dimensional reduction to four dimensions the isometry groups of the resulting homogeneous scalar manifolds extend to duality symmetry groups of the corresponding MESGTs. These isometry groups are not semisimple for homogeneous and

[20]We should note that in case the scalar manifold is a symmetric space the above Lie algebra gets extended by additional symmetry generators belonging to grade -1 space transforming in the conjugate representation of g_{+1} with respect to g_0. The generic non-Jordan family corresponds to theories with $q=-1$ and $\mathcal{S}(P)=\mathfrak{so}(P)$. On the other hand, for the generic Jordan family on has $q=0$ and $\mathcal{S}(P)=\mathfrak{so}(P)$. The magical theories correspond to $q=1,2,4,8$ and $P=1$ [37].

q	$\mathcal{C}(q+1,0)$	$\mathcal{D}_{q+1}$	$\mathcal{S}_q(P,Q)$
-1	$\mathbb{R}$	1	SO(P)
0	$\mathbb{R}\oplus\mathbb{R}$	1	SO$(P)\otimes$ SO(Q)
1	$\mathbb{R}(2)$	2	SO(P)
2	$\mathbb{C}(2)$	4	U(P)
3	$\mathbb{H}(2)$	8	USp$(2P)$
4	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	8	USp$(2P)\otimes$ USp$(2Q)$
5	$\mathbb{H}(4)$	16	USp$(2P)$
6	$\mathbb{C}(8)$	16	U(P)
7	$\mathbb{R}(16)$	16	SO(P)
$n+8$	$\mathbb{R}(16)\otimes\mathcal{C}(n+1,0)$	$16\ \mathcal{D}_n$	as for $q=n$

Table 5: Real Clifford algebras $\mathcal{C}(q+1,0)$ in $q+1$ euclidean dimensions. $\mathbb{A}(n)$ denotes the $n\times n$ matrices over the division algebra $\mathbb{A}$, while $\mathcal{D}_{q+1}$ denotes the real dimension of an irreducible representation of the Clifford algebra. The $\mathcal{S}_q(P,Q)$ is the metric preserving group in the centralizer of the Clifford algebra in the $(P+Q)\mathcal{D}_{q+1}$ dimensional representation.

non-symmetric scalar manifolds. Their Lie algebras $\mathfrak{W}$ have a graded decomposition of the form[21]

$$\mathfrak{W} = \mathfrak{W}^0 \oplus \mathfrak{W}^1 \oplus \mathfrak{W}^2 \tag{5.9}$$

with respect to the $\mathrm{SO}(1,1)_\alpha$ whose generator is

$$\alpha = \frac{2}{3}\lambda + \beta\,,$$

where λ is the generator of $\mathrm{SO}(1,1)_\lambda$ that determines the graded decomposition in five dimensions given above. For the 4D theories that come from the 5D theories listed in Table 5 the corresponding isometries are

$$\begin{aligned}
\mathfrak{W}^0 &= \mathfrak{so}(1,1)_\alpha \oplus \mathfrak{so}(q+2,2) \oplus \mathcal{S}_q(P,Q) \\
\mathfrak{W}^1 &= (1,S,V) \\
\mathfrak{W}^2 &= (2,0,0)\,,
\end{aligned}$$

where S is the spinor representation of $\mathrm{SO}(q+2,2)$ and V represents the $(P+Q)$ dimensional vector representation of $\mathcal{S}_q(P,Q)$. The grade $+2$ subspace is again one dimensional. Denoting the group generated by $\mathcal{W}$ as G one finds that the 4D scalar manifolds are the coset spaces

$$\mathcal{M}_4 = \frac{G}{H}\,,$$

where $H = \mathrm{SO}(q+2)\times\mathrm{SO}(2)\times\mathcal{S}_q(P,Q)$.[22]

[21] Again for theories whose scalar manifolds are symmetric spaces in four dimensions the full Lie algebras of isometries have , in addition, generators of grade -1 and -2, corresponding to hidden symmetries.

[22] Following [37] we label the group generated by $\mathcal{S}_q(P,Q)$ with the same symbol.

The maximal semisimple subgroups of the full isometry groups for homogeneous and nonsymmetric scalar manifolds in four dimensions is then of the form

$$\mathrm{SO}(q+2,2)\times\mathcal{S}_0(P,Q)\,,$$

where $\mathcal{S}_0(P,Q)$ denotes the semisimple subgroup of $\mathcal{S}(P,Q)$ listed in table 5.

In four dimensions we have $\tilde{n}+2=4+q+(P+Q)\mathcal{D}_{q+1}$ vector fields. Under the action of $\mathrm{SO}(q+2,2)\times\mathcal{S}(P,Q)$ the field strengths of these vector fields and their magnetic duals transform as follows

$$(\mathcal{F}^{A\,\mu\nu}\oplus\mathcal{G}_{A\,\mu\nu})\cong(q+4,1)+(q+4,1)+(S,P+Q)\,,$$

where $S=2\mathcal{D}_{q+1}$. Now $q\geq-1$ and for homogeneous and nonsymmetric scalar manifolds $P\geq1$ and $Q\geq1$ [37]. Hence in the corresponding theories the isometry groups do not have any simple subgroups under which all the vector field strengths transform irreducibly. Therefore they are not unified MESGTs.

5.4 Theories defined by lorentzian Jordan algebras

Let us now consider the 4D MESGTs defined by lorentzian Jordan algebras. The manifest isometries of the 4D scalar manifolds, $\mathcal{M}_4$, of these theories generate the semidirect product groups of the form

$$\left[\mathrm{Aut}\left(J^{\mathbb{A}}_{(1,N)}\right)\times\mathrm{SO}(1,1)\right]\circledS T_{J^{\mathbb{A}}_{(1,N)_0}}\,, \tag{5.10}$$

where $T_{J^{\mathbb{A}}_{(1,N)_0}}$ denotes the translations by traceless elements of $J^{\mathbb{A}}_{(1,N)}$. These isometries also form the invariance groups of the lagrangians of the corresponding MESGTs (modulo shifts in the theta angles). The Lie algebras of these symmetry groups have the graded decomposition

$$\mathfrak{g}=\mathfrak{g}^0\oplus\mathfrak{g}^1\,,$$

where the grading is with respect to the generator of scale transformations $\mathrm{SO}(1,1)$, as discussed in the previous section, and $\mathfrak{g}^0$ is the Lie algebra of $\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})\times\mathrm{SO}(1,1)$. An important question is whether or not there are further "hidden" isometries, in particular with negative scale dimension with respect to $\mathrm{SO}(1,1)$. For $\mathcal{N}=2$ MESGTs defined by lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,3)}$ of degree 4, where $\mathbb{A}=\mathbb{R},\mathbb{C}$ and $\mathbb{H}$ the cubic forms defined by the structure constants of the traceless elements coincide with the norm forms of euclidean Jordan algebras of degree 3 over $\mathbb{C},\mathbb{H}$ and $\mathbb{O}$, respectively. The isometry groups of the scalar manifolds of theories defined by euclidean Jordan algebras of degree 3 are simply the conformal groups of the corresponding euclidean Jordan algebras. Therefore in these three theories the Lie algebra of the isometry group has a 3-graded decomposition

$$\mathfrak{isom}(\mathcal{M}_4)=\mathfrak{K}_{J_3^{\mathbb{A}}}\oplus\mathfrak{str}(J_3^{\mathbb{A}})\oplus\mathfrak{T}_{J_3^{\mathbb{A}}}\,,$$

J	$\mathrm{Aut}(J)$	$\mathrm{Str}_0(J)$	$\mathrm{Conf}(J)$
$J^{\mathbb{R}}_{(1,3)}$	$\mathrm{SO}(3,1)$	$\mathrm{SL}(4,\mathbb{R})$	$\mathrm{Sp}(8,\mathbb{R})$
$J^{\mathbb{C}}_3$	$\mathrm{SU}(3)$	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SU}(3,3)$
$J^{\mathbb{C}}_{(1,3)}$	$\mathrm{SU}(3,1)$	$\mathrm{SL}(4,\mathbb{C})$	$\mathrm{SU}(4,4)$
$J^{\mathbb{H}}_3$	$\mathrm{USp}(6)$	$\mathrm{SU}^*(6)$	$\mathrm{SO}^*(12)$
$J^{\mathbb{H}}_{(1,3)}$	$\mathrm{USp}(6,2)$	$\mathrm{SU}^*(8)$	$\mathrm{SO}^*(16)$
$J^{\mathbb{O}}_3$	F_4	$E_{6(-26)}$	$E_{7(-25)}$

Table 6: List of the simple lorentzian Jordan algebras of type $J^{\mathbb{R}}_{(1,3)}, J^{\mathbb{C}}_{(1,3)}, J^{\mathbb{H}}_{(1,3)}$ and the corresponding euclidean Jordan algebras $J^{\mathbb{C}}_3, J^{\mathbb{H}}_3, J^{\mathbb{O}}_3$ and their automorphism, reduced structure and conformal (Möbius) groups.

where $\mathfrak{K}$ represent special conformal generators of negative dimension and $\mathfrak{T}$ denote translations. Since the cubic forms defined by the structure constants of the traceless elements of the lorentzian algebras $J^{\mathbb{A}}_{(1,3)}$ are not their generic norm forms, which are quartic, their conformal symmetry algebras are not the symmetries of the corresponding four dimensional MESGTs. To highlight the differences in the symmetry algebras of the lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,3)}$ and those of the corresponding euclidean Jordan algebras of degree three we list them in table 6.

For magical supergravity theories the global symmetry group of the five-dimensional theory is the reduced structure group $\mathrm{Str}_0(J)$ of the underlying Jordan algebra J and the number of vector fields $\tilde{n}$ is equal to the dimension $\dim(J)$ of J. Under dimensional reduction to 4D there is a vector field that is a singlet of $\mathrm{Str}_0(J)$, namely the vector field coming from the graviton in five dimensions. For these theories the full U-duality group is the conformal group of J under whose action electric and magnetic fields strengths transform in the irreducible $(2\tilde{n}+2)$ dimensional representation.

Viewing three of the magical theories as coming from the lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,3)}$ ($\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$), the manifest symmetry groups of the 5D theories are the automorphism groups of $J^{\mathbb{A}}_{(1,3)}$. However, they have "hidden" symmetries in five dimensions that extend the automorphism groups $J^{\mathbb{A}}_{(1,3)}$ ($\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$) to the reduced structure groups $\mathrm{Str}_0(J^{\mathbb{C}}_3)$, $\mathrm{Str}_0(J^{\mathbb{H}}_3)$ and $\mathrm{Str}_0(J^{\mathbb{O}}_3)$, respectively. By contrast, the reduced structure groups of the lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,3)}$ are not symmetry groups of the corresponding five-dimensional MESGTs.

The number of vector fields of the four-dimensional MESGTs defined by lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,3)}$ is equal to the dimension of $J^{\mathbb{A}}_{(1,3)}$. Under the action of their reduced structure groups the elements of $J^{\mathbb{A}}_{(1,3)}$ transform in real irreducible representations. As discussed above, there exist symplectic sections of the corresponding four-dimensional MESGTs such that the reduced structure groups of $J^{\mathbb{A}}_{(1,3)}$ become symmetries of their lagrangians. This is why they are unified MESGTs in four dimensions as well. The important fact to note is that the reduced structure groups of the lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,3)}$ are the maximal common subgroups of the conformal groups of $J^{\mathbb{A}}_{(1,3)}$ ($\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$) and of the conformal groups of the corresponding euclidean Jordan algebras $J^{\mathbb{A}'}_3$ ($\mathbb{A}'=\mathbb{C},\mathbb{H},\mathbb{O}$), respectively. The latter groups are the full U-duality groups of these theories in four dimensions.

For the five-dimensional $\mathcal{N}=2$ MESGTs defined by the lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ for $N\neq 3$, the scalar manifolds are not homogeneous. Hence the analysis of [37] regarding the existence of negative grade isometry generators does not apply. However, one can prove that, also for these theories, there cannot be any negative dimension isometry generators of the type that comes up in symmetric space theories as follows.

From the discussion in the previous section about the geometry of these theories it follows that the Lie algebra $[\mathfrak{Der}(J^{\mathbb{A}}_{(1,N)})\oplus\beta]\circledS\mathfrak{T}_{J_{(1,N)_0}}$ of the manifest symmetry group $[\mathrm{Aut}(J^{\mathbb{A}}_{(1,N)})\times \mathrm{SO}(1,1)]\circledS T_{J^{\mathbb{A}}_{(1,N)_0}}$ can be embedded in the natural 3-graded decomposition of the conformal algebra of $J^{\mathbb{A}}_{(1,N)}$ as follows:

$$\mathfrak{conf}\left(J^{\mathbb{A}}_{(1,N)}\right)=\left[\mathfrak{K}_I\oplus\mathfrak{K}_{J_{(1,N)_0}}\right]\oplus\left[\mathfrak{Der}(J^{\mathbb{A}}_{(1,N)})\oplus\beta\oplus\mathfrak{M}(J^{\mathbb{A}}_{(1,N)_0})\right]\oplus\left[\mathfrak{T}^{\mathbb{A}}_{J_{(1,N)_0}}+\mathfrak{T}_I\right], \tag{5.11}$$

where I denotes the identity element of the Jordan algebra and $\mathfrak{M}(J_0)$ denotes multiplications by traceless elements of J. The conformal Lie algebras of simple Jordan algebras of dimension greater than two are simple. From the commutation relations of the generators of conformal algebras of simple Jordan algebras [47], it follows that any single special conformal generator $\mathfrak{K}_a$ of negative dimension will extend the manifest symmetry algebra $[\mathfrak{Der}(J^{\mathbb{A}}_{(1,N)})\oplus\beta]\circledS\mathfrak{T}_{J_{(1,N)_0}}$ to the full conformal algebra $\mathfrak{conf}(J^{\mathbb{A}}_{(1,N)})$ under commutation. Now the number of vector fields in 4D theories defined by lorentzian Jordan algebras is equal to the dimension of the Jordan algebra itself:

$$l\equiv\dim(J^{\mathbb{A}}_{(1,N)})=N+1+\frac{1}{2}N(N+1)\dim(\mathbb{A})\,.$$

However, the conformal group of $J^{\mathbb{A}}_{(1,N)}$ acts nonlinearly on $J^{\mathbb{A}}_{(1,N)}$ and does not have any irreducible *linear* representation of dimension l, which decomposes into a singlet plus an irrep of dimension $(l-1)$ under its automorphism group. If the conformal group were to act as the full on-shell U-duality that transforms field strengths and their magnetic duals into each other, it would have to have a linear representation of dimension $2l$. This linear representation of dimension $2l$ must then decompose as two singlets and two irreps of dimension $(l-1)$ under the automorphism group of the Jordan algebra. Going through the list of the conformal groups of simple lorentzian Jordan algebras it is straightforward to show that they do not admit such a linear representation. Thus, for 4D, $\mathcal{N}=2$ MESGTs defined by the lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ their conformal groups can not be symmetries on-shell or off-shell. Hence, there cannot be any symmetry generators of negative dimension corresponding to special conformal transformations of the corresponding Jordan algebras.

Even though there cannot be any negative dimension hidden symmetry generators associated with conformal transformations in theories defined by lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ (for $N \neq 3$) in four dimensions, one may wonder if there could be extra hidden symmetry generators of dimension zero. In particular if extra symmetry generators corresponding to multiplications by traceless elements of the Jordan algebra were to arise as hidden symmetries under dimensional reduction they would extend the five-dimensional manifest symmetry group $\mathrm{Aut}(J_{(1,N)})$ to the reduced structure group that acts irreducibly on the Jordan algebra whose dimension l is equal to the number of vector fields in the theory. However the extension of the grade zero symmetry algebra to that of the reduced structure group would require that the number of translation generators be increased from $l-1$ to l (see appendix A). But we have only $l-1$ additional scalars that come from the vector fields in 5d. Therefore we can rule out the extension of the automorphism group to the reduced structure group within the symplectic section one obtains by dimensional reduction. However, in general the symplectic sections that make manifest a unifying symmetry group of the lagrangian break the translational symmetry. For example, in the exceptional supergravity theory the unifying symmetry group is $\mathrm{SU}^*(8)$ under which the remaining generators of the full duality group $E_{7(-25)}$ transform in the 70 dimensional representation, whose action does not leave the lagrangian invariant! Therefore, for the $\mathcal{N}=2$ MESGTs defined by lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ $(N \neq 3)$, we can not rule out the existence of symplectic sections where their reduced structure groups become unifying symmetry groups of their lagrangians. What makes the analysis of these theories more complicated is the fact that their scalar manifolds are not homogeneous. We leave this as an open problem for future investigations.

We finally note that, under dimensional reduction to 4D, one gets an extra vector multiplet from the 5 dimensional graviton supermultiplet. One may ask if it possible to obtain unified MESGTs in 4D by first dimensionally reducing the 5d unified MESGTs to four dimensions followed by a truncation of this extra vector supermultiplet. As we discussed at the beginning of this section the scalars of this extra vector supermultiplet parameterize the manifold

$$\frac{\mathrm{SU}(1,1)_G}{\mathrm{U}(1)}\,.$$

For the theories defined by euclidean Jordan algebras we listed the centralizers of the group $\mathrm{SU}(1,1)_G$ in section 5.2. After truncation the centralizing groups would become the U-duality groups. It is manifest from the list of centralizing groups that, even if the truncations are consistent, the resulting theories can not be unified since there will always be a singlet vector field of the resulting U-duality groups. The same analysis can be extended to the theories defined by lorentzian Jordan algebras of degree greater than four. In this case the $\mathrm{SU}(1,1)_G$ is not part of the full U-duality group in four dimensions, but rather its parabolic subgroup $\mathrm{SO}(1,1)$ⓈT is. It can be shown that the centralizer of this parabolic subgroup will have a singlet vector field after the truncation, even if the truncation is consistent.

6. Conclusion and summary

In this paper, we studied the four-dimensional unified $\mathcal{N}=2$ MESGTs that can be obtained by dimensional reduction from 5D and their possible gaugings so as to obtain unified YMESGTs. We found that the generic Jordan family of MESGTs that are not unified in 5D become unified in 4D. The electric subgroup of the full duality group is $\mathrm{SO}(\tilde{n},2)$ under which all the vector field strengths, including that of the graviphoton, transform irreducibly. Of these theories only one of them (corresponding to $\tilde{n}=1$) can be gauged so as to obtain a unified YMESGT with the gauge group $\mathrm{SO}(2,1)=\mathrm{SU}(1,1)$. This gauging breaks the full duality group $\mathrm{SO}(2,1)\times\mathrm{SO}(2,1)$ down to $\mathrm{SO}(2,1)$. The four magical supergravity theories defined by simple Jordan algebras of degree three are unified MESGTs in 5D. In 4D they all have simple duality groups. However only the theories defined by $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ are unified MESGTs in 4D , with unifying symmetry groups $\mathrm{SL}(4,\mathbb{R}),\mathrm{SL}(4,\mathbb{C})$ and $\mathrm{SU}^*(8)$, respectively. The $\mathrm{Sp}(4,\mathbb{R})=\mathrm{SO}(3,2)$ subgroup of the full duality group $\mathrm{SU}(3,3)$ of the $J_3^{\mathbb{C}}$ theory can ge gauged so as to obtain a unified YMESGT in 4D. The theory defined by $J_3^{\mathbb{H}}$ can not be gauged so as to obtain a unified YMESGT. In the exceptional supergravity defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ one can gauge the $\mathrm{SO}^*(8)=\mathrm{SO}(6,2)$ subgroup of the duality group $E_{7(-25)}$ to get a unified YMESGT in 4D. These results are summarized in Table 4.

The generic non-Jordan family and the theories whose scalar manifolds are homogeneous but not symmetric do not lead to unified MESGTs in 4D.

The unified five-dimensional MESGTs defined by lorentzian Jordan algebras, $J^{\mathbb{A}}_{(1,N)}$, of degree $p=(N+1)$, $(N\neq 3)$ do not have hidden symmetries associated with special conformal transformations of $J^{\mathbb{A}}_{(1,N)}$. Whether they admit other hidden symmetries that admit symplectic sections with their reduced structure groups acting as their unifying symmetry groups is an open problem.

In this paper, we considered the 4D, $\mathcal{N}=2$ MESGTs that originate from five dimensions. We should perhaps mention that there is a family of 4D MESGTs whose scalar manifolds are the symmetric spaces [48]

$$\frac{\mathrm{SU}(n,1)}{\mathrm{U}(n)}$$

which do not originate from five dimensions. They are unified MESGTs since all the vector field strengths transform irreducibly under the electric subgroup $\mathrm{SO}(n,1)$. Of these only the theory with $n=2$ can be gauged so as to obtain a unified YMESGT with the gauge group $\mathrm{SO}(2,1)$. This unified YMESGT is different from the $\mathrm{SO}(2,1)$ gauged unified YMESGT considered in section 5, since their scalar manifolds are different, namely one has

$$\mathrm{SU}(2,1)/\,\mathrm{U}(2)$$

and

$$\frac{\mathrm{SU}(1,1)\times\mathrm{SO}(2,1)}{\mathrm{U}(1)\times\mathrm{U}(1)}\,.$$

A. Symmetry algebras of Jordan algebras

Jordan algebras are commutative

$$a \circ b = b \circ a$$

and, in general, non-associative algebras that satisfy the identity

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2)\,.$$

Elements of a Jordan algebra form a so-called Jordan triple system (JTS) under the Jordan triple product

$$\{a\,b\,c\} = a \circ (b \circ c) + (a \circ b) \circ c - b \circ (a \circ c)\,.$$

This triple product satisfies the identities

$$\begin{aligned} &\{a\,b\,c\} = \{c\,b\,a\}\,, \\ &\{a\,b\,\{c\,d\,x\}\} - \{c\,d\,\{a\,b\,x\}\} - \{a\,\{d\,c\,b\}\,x\} + \{\{c\,d\,a\}\,b\,x\} = 0\,, \end{aligned}$$

which are the defining identities of a Jordan triple system.

The Lie algebra $\mathfrak{aut}(J)$ of the automorphism group of a Jordan algebra J is simply its derivation algebra $\mathrm{Der}(J)$. Derivations are defined as linear transformations D that satisfy the Leibniz rule; i.e., given $a, b \in J$

$$D(a \circ b) = (Da) \circ b + a \circ (Db) \in J\,.$$

They close and form a Lie algebra under commutator product. Derivations of a Jordan algebra can always be written in the form [29]

$$D_{a,b} \equiv [M_a, M_b]\,; \qquad a, b \in J\,,$$

where M_a denotes multiplication by element a:

$$M_a x = a \circ x\,.$$

Clearly the multiplications M_J close into derivations under commutation. The Lie algebra $\mathfrak{str}(J)$ of the structure group of J is then generated by derivations and multiplications $\mathrm{M}(J)$ by the elements of J

$$\mathfrak{str}(J) = \mathrm{Der}(J) \oplus \mathrm{M}(J)\,.$$

Multiplications by *traceless* elements of J together with derivations generate the reduced structure group $\mathrm{Str}_0(J)$, which turns out to be the invariance group of norm of the Jordan algebra J.

For a euclidean Jordan algebra derivations generate a compact automorphism group and the multiplications $M(J)$ by the elements of J correspond to non-compact generators of $\mathfrak{str}(J)$. For a noncompact, e.g., lorentzian, Jordan algebra the automorphism group is noncompact and hence some of the derivations correspond to noncompact generators. Similarly multiplications M_a by an element $a \in J$ will generate compact or noncompact transformation depending on whether the element a is anti-hermitean or hermitean, respectively. Even though the automorphism groups of euclidean Jordan algebras $J^{\mathbb{A}}_{(N+1)}$ and lorentzian Jordan algebras $J^{\mathbb{A}}_{(1,N)}$ are not isomorphic, their structure algebras are:

$$\mathfrak{str}(J^{\mathbb{A}}_{(N+1)}) \approx \mathfrak{str}(J^{\mathbb{A}}_{(1,N)}) .$$

Koecher [49] introduced the concept of linear fractional transformation groups of Jordan algebras. The linear fractional groups can be interpreted as generalized conformal groups of spacetimes coordinatized by Jordan algebras [50], which we denote as Conf(J). Their Lie algebras can be given a 3-graded decomposition [51]

$$\mathfrak{conf}(J) = g^{-1} \oplus g^{0} \oplus g^{+1} = K_J \oplus \mathfrak{str}(J) \oplus T_J ,$$

where the grade zero subalgebra is simply the structure algebra and grade +1 and -1 subspaces are spanned by translations T_J and special conformal generators K_J, respectively. Within this framework the reduced structure group is simply the generalized Lorentz group Lor(J) of the spacetime defined by J [50], and, in addition, the structure group includes the dilatations.

The Tits-Kantor-Koecher (TKK) construction [51] associates with every JTS a 3-graded Lie algebra[23]

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} ,$$

In the TKK construction the elements of the $\mathfrak{g}^{+1}$ subspace of the Lie algebra $\mathfrak{g}$ are labeled by the elements $a \in J$:

$$U_a \in \mathfrak{g}^{+1} \Leftrightarrow a \in J .$$

Furthermore every such Lie algebra $\mathfrak{g}$ admits an involution $\tilde{}$, which maps the elements of the grade +1 space onto the elements of the subspace of grade -1:

$$\tilde{} : U_a \in \mathfrak{g}^{+1} \Rightarrow \tilde{U}_a \in \mathfrak{g}^{-1} .$$

To get the complete set of generators of $\mathfrak{g}$ one defines further

$$\begin{aligned} [U_a, \tilde{U}_b] &=: S_{ab} \\ [S_{ab}, U_c] &=: U_{\{abc\}} , \end{aligned}$$

where $S_{ab} \in \mathfrak{g}^0$ and $\{abc\}$ is the Jordan triple product under which J is closed.

The remaining commutation relations are

$$\begin{aligned} [S_{ab}, \tilde{U}_c] &= \tilde{U}_{\{bac\}} \\ [S_{ab}, S_{cd}] &= S_{\{abc\}d} - S_{c\{bad\}} , \end{aligned}$$

and the Jacobi identities follow from the defining identities of a JTS given above.

[23]With the exception of the Lie algebras of G_2, F_4 and E_8 every simple Lie algebra $\mathfrak{g}$ can be given a three graded decomposition with respect to a subalgebra $\mathfrak{g}^0$ of maximal rank.

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The Lie algebra generated by S_{ab} is simply the structure algebra of J since

$$S_{ab} = D_{a,b} + M_{a\circ b}\,,$$

where $D_{a,b} = [M_a, M_b]$ is a derivation and $M_{a\circ b}$ is multiplication by the element $a \circ b$. The traceless elements of this action of S_{ab} generate the reduced structure algebra of J.

The above concepts are best illustrated in terms of a simple and familiar example, namely the conformal group in four dimensions, and its realization via the Jordan algebra $J_2^{\mathbb{C}}$ of hermitean 2×2 matrices over $\mathbb{C}$. As is well known, these matrices are in one-to-one correspondence with four-vectors x^μ in Minkowski space

$$x \equiv x_\mu \sigma^\mu \in J_2^{\mathbb{C}}\,,$$

where $\sigma^\mu := (1, \vec{\sigma})$. The "norm form" on this Jordan algebra is just the ordinary determinant, i.e.

$$N(x) := \det(x) = x_\mu x^\mu\,.$$

The automorphism group of $J_2^{\mathbb{C}}$ is just the rotation group SU(2), its structure group is the product $\mathrm{SL}(2, \mathbb{C}) \times \mathcal{D}$, i.e. the Lorentz group times dilatations. The conformal group of $J_2^{\mathbb{C}}$ is $su(2, 2)$, which has a three-graded structure

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1}\,,$$

where the grade $+1$ and grade -1 spaces correspond to generators of translations P^μ and special conformal transformations K^μ, respectively, while the grade 0 subspace is spanned by the Lorentz generators $M^{\mu\nu}$ and the dilatation generator D. The subspaces $\mathfrak{g}^{+1}$ and $\mathfrak{g}^{-1}$ can both be labelled by the elements of the Jordan algebra $J_2^{\mathbb{C}}$. Letting $a = a_\mu \sigma^\mu$ and $b = b_\mu \sigma^\mu$ we can expand the grade +1 and -1 subspaces in terms of the generators of translations P_μ and special conformal transformations K_μ as

$$\begin{aligned} U_a &:= a_\mu P^\mu \in \mathfrak{g}^{+1} \\ \tilde{U}_b &:= b_\mu K^\mu \in \mathfrak{g}^{-1}\,. \end{aligned}$$

The generators in $\mathfrak{g}^0$ are labeled as

$$S_{ab} := a_\mu b_\nu (M^{\mu\nu} + \eta^{\mu\nu} D)\,.$$

The conformal group is realized non-linearly on the space of four-vectors $x \in J_2^{\mathbb{C}}$, with a Möbius-like action, which acquires a very simple form when expressed in terms of the Jordan triple product, namely

$$\begin{aligned} U_a(x) &= a \\ S_{ab}(x) &= \{a\,b\,x\} \\ \tilde{U}_c(x) &= -\frac{1}{2}\{x\,c\,x\}\,, \end{aligned}$$

Such a realization extends to all JTS's defined by Jordan algebras [50].

B. Symplectic formulation of $N = 2$ MESGTs in four dimensions

The self-dual field strengths $F^{A+}_{\mu\nu}$ and their magnetic "duals" [52, 53],

$$G^{+}_{\mu\nu A} := \frac{\delta\mathcal{L}^{(4)}}{\delta F^{A+}_{\mu\nu}} = -\frac{i}{2}\mathcal{N}_{AB}F^{\mu\nu B+}\,, \tag{B.1}$$

can be combined into a symplectic vector

$$\begin{pmatrix} F^{A+}_{\mu\nu} \\ G^{+}_{\mu\nu B} \end{pmatrix} \tag{B.2}$$

so that the equations of motion that follow from (4.16) are invariant under the (global) symplectic rotations

$$\begin{pmatrix} X^{A} \\ F_{B} \end{pmatrix} \longrightarrow \mathcal{O}\begin{pmatrix} X^{A} \\ F_{B} \end{pmatrix}, \qquad \begin{pmatrix} F^{A+}_{\mu\nu} \\ G^{+}_{\mu\nu B} \end{pmatrix} \longrightarrow \mathcal{O}\begin{pmatrix} F^{A+}_{\mu\nu} \\ G^{+}_{\mu\nu B} \end{pmatrix} \tag{B.3}$$

with $\mathcal{O}$ being a symplectic matrix with respect to the symplectic metric

$$\omega = \begin{pmatrix} 0 & \delta^{A}_{B} \\ -\delta^{A}_{B} & 0 \end{pmatrix}. \tag{B.4}$$

Writing $\mathcal{O}$ as

$$\mathcal{O} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{B.5}$$

the period matrix $\mathcal{N}$ transforms as

$$\mathcal{N} \longrightarrow (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\,. \tag{B.6}$$

Symplectic transformations with $B \neq 0$ correspond to electromagnetic duality transformation, whereas transformations with $C \neq 0$ involve shifts of the theta angles in the lagrangian.

General symplectic tranformations will take a lagrangian $\mathcal{L}(F, G)$ with the field strengths satisfying the Bianchi identities $dF^{A} = 0$ and $dG_{A} = 0$ to a lagrangian $\tilde{\mathcal{L}}(\tilde{F}, \tilde{G})$ with the new field strengths satisfying $d\tilde{F}^{A} = 0$ and $d\tilde{G}_{A} = 0$. Theories related by such general symplectic tranformations are called dual theories.

The subgroup, $\mathcal{U}$, of $\mathrm{Sp}(2(\tilde{n}+2), \mathbb{R})$ that leaves the functional invariant [37]

$$\tilde{\mathcal{L}}(\tilde{F}, \tilde{G}) = \mathcal{L}(\tilde{F}, \tilde{G})\,,$$

is called the duality invariance group (or "U-duality group"). This is a symmetry group of the on-shell theory (i.e., of the equations of motion). A subgroup of the duality invariance group that leaves the off-shell lagrangian invariant up to surface terms is called an "electric subgroup", G_{E}, since it transforms electric field strengths into electric field strengths only. Obviously, we have the inclusions

$$G_{E} \subset \mathcal{U} \subset \mathrm{Sp}(2(\tilde{n}+2), \mathbb{R})\,,$$

and the pure electric-magnetic transformations are contained in the coset $\mathcal{U}/G_E$, while dualizations leading to different manifest electric subgroups G_E groups are contained in $\mathrm{Sp}(2n+4)/\mathcal{U}$ [5].

For the symplectic formulation of $\mathcal{N}=2$ MESGTs in 4D and further references we refer the reader to [54].

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